Vertex alignment and changepoint localization in network time series

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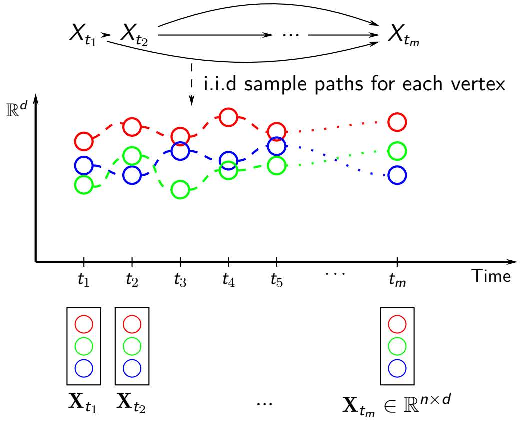
In this talk, we will discuss:

- The *latent position process time series of graphs* (LPPTSG): a generative model for a time series of graphs, governed by an underlying stochastic process *that can induce time dependence for each vertex*
- The *Euclidean mirror*: a way to represent dynamics in time series of graphs, with use in changepoint localization for network time series
- How *vertex misalignment* affects the Euclidean mirror
- *Two specific latent position processes*: *London* and *Atlanta*, and their associated network time series
- The impact of *vertex misalignment for changepoint localization* in both the London and Atlanta models

The latent position process: the *DNA* of a class of network time series

Latent position process (LPP):



Latent position matrices

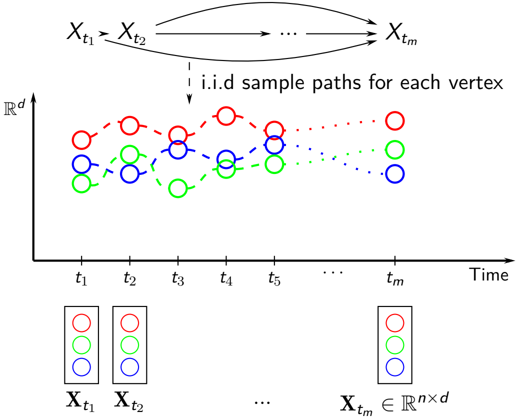
For time t, rows give latent positions for each vertex:

$$(\mathbf{X}_t)_1, (\mathbf{X}_t)_2, ..., (\mathbf{X}_t)_n \overset{\text{i.i.d}}{\sim} \mu_{X_t} \;;$$

For vertex i across times $t_1, \cdots, t_m$, joint distribution of

$$(\mathbf{X}_{t_1})_i, (\mathbf{X}_{t_2})_i, ..., (\mathbf{X}_{t_m})_i \sim \mu_{X_{t_1}, X_{t_2}, ..., X_{t_m}}.$$

Latent position process (LPP):



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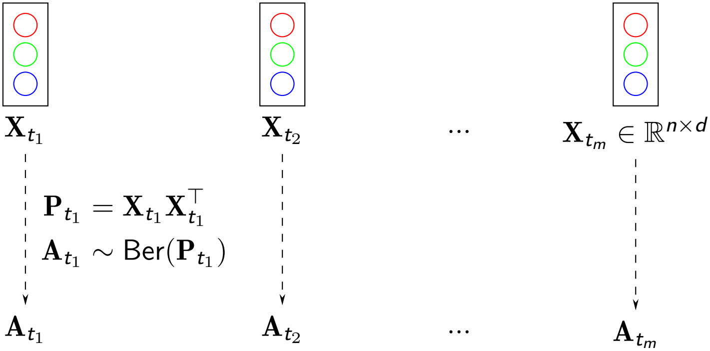
For vertex i across times $t_1, \cdots, t_m$, joint distribution of

$$(\mathbf{X}_{t_1})_i, (\mathbf{X}_{t_2})_i, ..., (\mathbf{X}_{t_m})_i \sim \mu_{X_{t_1}, X_{t_2}, ..., X_{t_m}}.$$

Under vertex misalignment: $(\mathbf{X}_{t_1})_{i}, (\mathbf{X}_{t_2})_{\sigma_2(i)}, ..., (\mathbf{X}_{t_m})_{\sigma_m(i)} \sim ???.$

Latent position processes and the generation of time series of graphs (LPPTSG)



The LPP influences the dynamics of the TSG.

The Euclidean mirror: a low-dimensional representation of network dynamics

The *Euclidean mirror* is based on *pairwise dissimilarity* for the LPP at different time points:

LPP: X_{t_1} X_{t_2} $\cdots$ X_{t_m}

Dissimilarity: $d_{MV}(X, Y) := \min_{W \in \mathcal{O}^{d \times d}} \left\| \mathbb{E}[(X - WY)(X - WY)^\top] \right\|_2^{1/2}$

a *quasi-covariance*, based on the joint distribution of (X, Y)

$$\begin{pmatrix} 0 & d_{MV}(X_{t_1}, X_{t_2}) & \cdots & d_{MV}(X_{t_1}, X_{t_m}) \\ d_{MV}(X_{t_2}, X_{t_1}) & 0 & \cdots & d_{MV}(X_{t_2}, X_{t_m}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{MV}(X_{t_m}, X_{t_1}) & d_{MV}(X_{t_m}, X_{t_2}) & \cdots & 0 \end{pmatrix}_{m \times m}$$

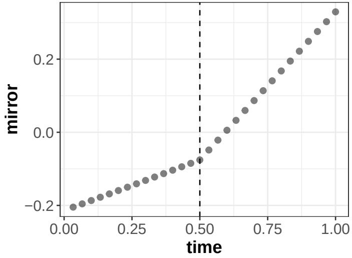
classical multidimensional scaling

$\psi \in \mathbb{R}^{m \times c}$ Euclidean mirror

Example: Euclidean mirror for changepoint localization

In the figure below:

- we plot the Euclidean mirror for a particular LPP TSG
- the LPP itself has an underlying changepoint at $t^* = 0.5$
- the associated Euclidean mirror has a slope change at t^*



Estimating the Euclidean mirror [ALPP25]

$\mathbf{A}_1 \quad \mathbf{A}_2 \quad \cdots \quad \mathbf{A}_m$

$\mathbf{A}_1 \approx \mathbf{U}_d \Sigma_d \mathbf{U}_d^\top$ ASE ASE

$\hat{\mathbf{X}}_1 = \mathbf{U}_d \Sigma_d^{1/2} \quad \hat{\mathbf{X}}_2 \quad \cdots \quad \hat{\mathbf{X}}_m$

Define $\hat{d}_{MV}(\hat{\mathbf{X}}_t, \hat{\mathbf{X}}_s) := \min_{W \in \mathcal{O}^{d \times d}} \frac{1}{\sqrt{n}} \|\hat{\mathbf{X}}_t - \hat{\mathbf{X}}_s W\|_2.$

known 1-1 vertex correspondence

$$\begin{pmatrix} 0 & \hat{d}_{MV}(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2) & \cdots & \hat{d}_{MV}(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_m) \\ \hat{d}_{MV}(\hat{\mathbf{X}}_2, \hat{\mathbf{X}}_1) & 0 & \cdots & \hat{d}_{MV}(\hat{\mathbf{X}}_2, \hat{\mathbf{X}}_m) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{d}_{MV}(\hat{\mathbf{X}}_m, \hat{\mathbf{X}}_1) & \hat{d}_{MV}(\hat{\mathbf{X}}_m, \hat{\mathbf{X}}_2) & \cdots & 0 \end{pmatrix}_{m \times m}$$

classical multidimensional scaling

$\hat{\psi} \in \mathbb{R}^{m \times c}$ estimated Euclidean mirror

Estimating the Euclidean mirror [ALPP25]

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Define $\hat{d}_{MV}(\hat{\mathbf{X}}_t, \hat{\mathbf{X}}_s) := \min_{W \in \mathcal{O}^{d \times d}} \frac{1}{\sqrt{n}} \|\hat{\mathbf{X}}_t - \hat{\mathbf{X}}_s W\|_2.$

$$\hat{d}_{MV}(\hat{\mathbf{X}}_t, \hat{\mathbf{X}}_s) \xrightarrow{p} d_{MV}(X_t, X_s)$$

$$\begin{pmatrix} 0 & \hat{d}_{MV}(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2) & \cdots & \hat{d}_{MV}(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_m) \\ \hat{d}_{MV}(\hat{\mathbf{X}}_2, \hat{\mathbf{X}}_1) & 0 & \cdots & \hat{d}_{MV}(\hat{\mathbf{X}}_2, \hat{\mathbf{X}}_m) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{d}_{MV}(\hat{\mathbf{X}}_m, \hat{\mathbf{X}}_1) & \hat{d}_{MV}(\hat{\mathbf{X}}_m, \hat{\mathbf{X}}_2) & \cdots & 0 \end{pmatrix}_{m \times m}$$

classical multidimensional scaling

$\hat{\psi} \in \mathbb{R}^{m \times c}$ estimated Euclidean mirror

Estimating the Euclidean mirror [ALPP25]

$$\begin{array}{cccc} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_m \\ \downarrow \ \mathbf{A}_1 \approx \mathbf{U}_d \Sigma_d \mathbf{U}_d^\top & \downarrow \text{ASE} & & \downarrow \text{ASE} \\ \hat{\mathbf{X}}_1 = \mathbf{U}_d \Sigma_d^{1/2} & \hat{\mathbf{X}}_2 & \cdots & \hat{\mathbf{X}}_m \end{array}$$

Define $\hat{d}_{MV}(\hat{\mathbf{X}}_t, \hat{\mathbf{X}}_s) := \min_{W \in \mathcal{O}^{d \times d}} \frac{1}{\sqrt{n}} \|\hat{\mathbf{X}}_t - \hat{\mathbf{X}}_s W\|_2$.

$\hat{d}_{MV}(\hat{\mathbf{X}}_t, P\hat{\mathbf{X}}_s) \xrightarrow{p} ?$

$$\downarrow$$

$$\begin{pmatrix} 0 & \hat{d}_{MV}(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2) & \cdots & \hat{d}_{MV}(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_m) \\ \hat{d}_{MV}(\hat{\mathbf{X}}_2, \hat{\mathbf{X}}_1) & 0 & \cdots & \hat{d}_{MV}(\hat{\mathbf{X}}_2, \hat{\mathbf{X}}_m) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{d}_{MV}(\hat{\mathbf{X}}_m, \hat{\mathbf{X}}_1) & \hat{d}_{MV}(\hat{\mathbf{X}}_m, \hat{\mathbf{X}}_2) & \cdots & 0 \end{pmatrix}_{m \times m}$$

$\downarrow$ classical multidimensional scaling

$\hat{\psi} \in \mathbb{R}^{m \times c}$ estimated Euclidean mirror

Understanding vertex misalignment

We start with an LPP time series of graphs where *vertex alignment is known*.

$$\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_m.$$

What is the impact of *vertex misalignment across time* on subsequent inference?

- Let $\{\sigma_1, \sigma_2, ...\sigma_m\}$ be independent permutations of the vertices, with associated permutation matrices $\{P_1, P_2, ..., P_m\}$.
- The adjacency matrices of the *shuffled TSG* are

$$P_1\mathbf{A}_1P_1^\top \qquad P_2\mathbf{A}_2P_2^\top \qquad \cdots \qquad P_m\mathbf{A}_mP_m^\top$$

What happens if we apply the Euclidean mirror to the shuffled TSG?

Understanding vertex misalignment

We start with an LPP time series of graphs where *vertex alignment is known*.

$$\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_m.$$

We want to understand the impact of *vertex misalignment across time* on subsequent inference

- Let $\{\sigma_1, \sigma_2, ...\sigma_m\}$ be independent permutations of the vertices, with associated permutation matrices $\{P_1, P_2, ..., P_m\}$.
- The adjacency matrices of the *shuffled TSG* are

$$\begin{array}{cccc} P_1\mathbf{A}_1P_1^\top & P_2\mathbf{A}_2P_2^\top & \cdots & P_m\mathbf{A}_mP_m^\top \\ \downarrow \text{ASE} & \downarrow \text{ASE} & & \downarrow \text{ASE} \\ P_1\hat{\mathbf{X}}_1 & P_2\hat{\mathbf{X}}_2 & \cdots & P_m\hat{\mathbf{X}}_m \end{array}$$

Vertex misalignment, continued

What happens if we apply the Euclidean mirror to the shuffled TSG?

First, consider the case when the true latent positions are known. Apply $\hat{d}_{MV}$ on shuffled true latent position matrices:

$$\begin{aligned}\hat{d}_{MV}(P_t\mathbf{X}_t, P_s\mathbf{X}_s) &:= \min_{W\in\mathcal{O}^{d\times d}} \frac{1}{\sqrt{n}}\|P_t\mathbf{X}_t - P_s\mathbf{X}_s W\|_2 \\ &= \min_{W\in\mathcal{O}^{d\times d}} \frac{1}{\sqrt{n}}\|\mathbf{X}_t - P_t^{-1}P_s\mathbf{X}_s W\|_2.\end{aligned}$$

Vertex misalignment and associated latent positions

$\|\mathbf{X}_t - P_t^{-1} P_s \mathbf{X}_s\|_2$ is based on the *shuffled true latent positions*:

$$\begin{pmatrix} (\mathbf{X}_t)_1, & (\mathbf{X}_s)_1 \\ (\mathbf{X}_t)_2, & (\mathbf{X}_s)_2 \\ (\mathbf{X}_t)_3, & (\mathbf{X}_s)_3 \\ \vdots & \vdots \\ (\mathbf{X}_t)_{10}, & (\mathbf{X}_s)_{10} \end{pmatrix} \xrightarrow{\text{shuffle}} \begin{pmatrix} (\mathbf{X}_t)_1, & (\mathbf{X}_s)_1 \\ (\mathbf{X}_t)_2, & (\mathbf{X}_s)_3 \\ (\mathbf{X}_t)_3, & (\mathbf{X}_s)_5 \\ \vdots & \vdots \\ (\mathbf{X}_t)_{10}, & (\mathbf{X}_s)_7 \end{pmatrix}$$

$$\left\{ ((\mathbf{X}_t)_i, (\mathbf{X}_s)_i)\, ; i \in [n] \right\} \qquad \left\{ \left((\mathbf{X}_t)_i, (\mathbf{X}_s)_{\sigma(i)}\right) ; i \in [n] \right\}$$

pairs are i.i.d from LPP

no longer i.i.d

Shuffling weakens dependence

$\{(x_i, y_i); i \in [n]\}$ are i.i.d samples drawn from a joint distribution $\mu_{X,Y}$. σ is a uniform random permutation that is independent of the sampled latent position process.

What is the empirical distribution of $\{(x_i, y_{\sigma(i)}); i \in [n]\}$?

$$\text{when} \quad \sigma(i) = i, \quad (x_i, y_{\sigma(i)}) \sim \mu_{X,Y}.$$

$$\text{when} \quad \sigma(i) \neq i, \quad x_i \perp\!\!\!\perp y_{\sigma(i)} \implies (x_i, y_{\sigma(i)}) \sim \mu_X \otimes \mu_Y.$$

$$\mathbb{E}[\#\{i : \sigma(i) = i\}] = 1 \ \ \forall \ n.$$

$$\mathbb{E}[\#\{i : \sigma(i) \neq i\}] = n - 1 \ \ \forall \ n.$$

Shuffling weakens dependence

Theorem (Shuffling weakens dependence)

For every function $f : \mathbb{R}^2 \to \mathbb{R}^1$ bounded on the support of $\mu_{X,Y}$, denoted as $\mathcal{S}$, and $\forall \varepsilon > 0$,

$$\mathbb{P}\left(\left| \frac{1}{n} \sum_{i=1}^{n} f\left(x_i, y_{\sigma(i)}\right) - \mathbb{E}_{\mu_X \otimes \mu_Y}[f] \right| > \varepsilon \right) \leq \frac{12 B_{f,\mathcal{S}}^2}{n\varepsilon^2} + \frac{4 B_{f,\mathcal{S}}}{n\varepsilon}.$$

Shuffling weakens dependence

Q: What information is lost when we lose the vertex correspondence?

A: We lose the information in the joint distribution $\mu_{X,Y}$ and retain information in the marginal distributions μ_X, μ_Y.

Shuffled-d_{MV}

Theorem (Shuffling vertices yields a shuffled-d_{MV} dissimilarity)

Consider a class of 1-d LPPs in which the d_{MV} distance for any two times simplifies to:

$$d^2_{MV}(X_t, X_{t'}) = \mathbb{E}[(X_t - X_{t'})^2], \qquad \hat{d}^2_{MV}(\mathbf{X}_t, \mathbf{X}_{t'}) = \frac{1}{n}\|\mathbf{X}_t - \mathbf{X}_{t'}\|_F^2.$$

then for any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\left|\hat{d}^2_{MV}(\mathbf{X}_t, P_\sigma \mathbf{X}_s) - \textit{shuffled-}d^2_{MV}(X_t, X_s)\right| \geq \varepsilon\right) \leq \frac{12}{n\varepsilon^2} + \frac{4}{n\varepsilon}.$$

Shuffled-d_{MV}

$$\text{shuffled-}d_{MV}\left(X_t, X_{t'}\right) := \min_{W \in \mathcal{O}^{d \times d}} \mathbb{E}\left\| \left(X'_t - WX'_{t'}\right)\left(X'_t - WX'_{t'}\right)^\top \right\|_2^{1/2},$$

where

$$X'_t \overset{\mathcal{L}}{=} X_t, \quad X'_{t'} \overset{\mathcal{L}}{=} X_{t'}, \quad \text{and} \quad X'_t \perp\!\!\!\perp X'_{t'}.$$

Note: ***shuffled-d_{MV} is wholly determined by the marginal distributions of $(X_t, X_{t'})$—not their joint distribution.***

Two edge-case LPPs, London and Atlanta.

Goal:

1. To understand the impact of vertex misalignment on the Euclidean mirror
2. More specifically, understand d_{MV} and shuffled-d_{MV}

Both models have a changepoint t^*.

London: shuffled-$d_{MV} \approx d_{MV}$ and estimation of t^* is *robust to misalignment.*

Atlanta: shuffled-$d_{MV} \neq d_{MV}$ and estimation of t^* is *sensitive to misalignment.*

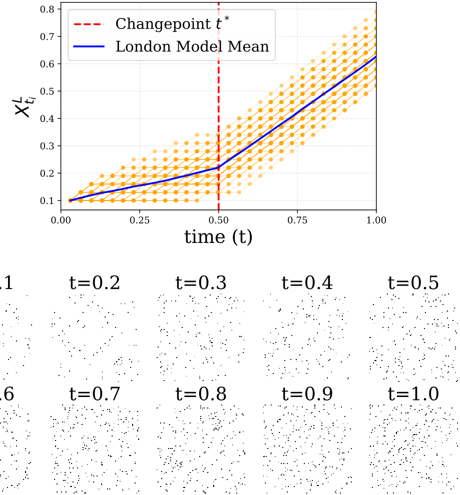
The *London* latent position process

$$X_0^L = 0 \quad \text{with probability } 1,$$

$$X_i^L = \begin{cases} X_{i-1}^L + \frac{1}{m} & \text{with probability } p \\ X_{i-1}^L & \text{with probability } 1 - p. \end{cases}$$

Jump probability p will change to q after t^*.

London LPP and TSG; adequate signal for changepoint localization



The *Atlanta* latent position process

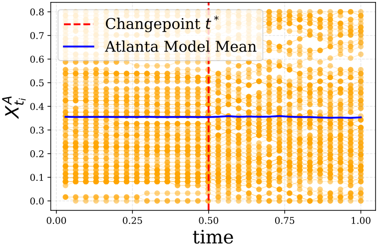
Let u denote the uniform distribution on the discrete set $\{0, \frac{1}{N-1}, \frac{2}{N-1}, \dots, 1\}$.

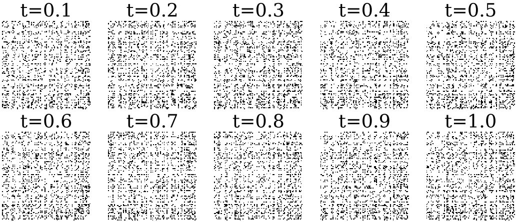
$$X_0^A \sim u,$$

$$X_i^A = \begin{cases} X_{i-1}^A + \frac{1}{N-1} & \text{with probability } p, \\ X_{i-1}^A - \frac{1}{N-1} & \text{with probability } p, \\ X_{i-1}^A & \text{with probability } 1 - 2p, \end{cases}$$

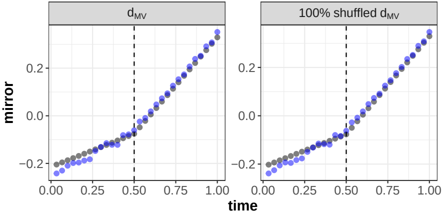
jump probability will change from p to q at t^*.

Atlanta LPP and TSG: loss of signal for changepoint localization

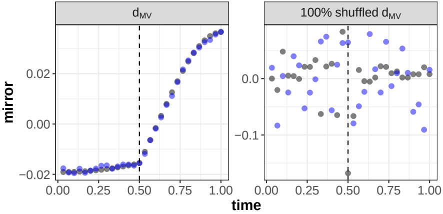




Mirror and shuffled mirror for both models



London model: $n = 100$, $p = 0.3$, $q = 0.9, m = 30$, $t^* = 0.5$.



Atlanta model: $n = 1000$, $p = 0.05$, $q = 0.45$, $m = 30$, $t^* = 0.5$.

To summarize, we discussed

- *Latent position process time series of graphs* (LPPTSG)
- The *Euclidean mirror*, changepoint localization, and time-dependence in the LPP
- The impact of *vertex misalignment* on the Euclidean mirror
- *Two specific latent position processes*: *London* and *Atlanta*, as well as their associated network time series, that serve as illustrative edge-cases for understanding vertex alignment in multiple network inference
- The London LPP TSG, which is robust to vertex misspecification, and the Euclidean mirror can still recover the changepoint even after shuffling
- The Atlanta LPP TSG, in which the Euclidean mirror can recover the changepoint when misalignment is not severe, but it *loses structure and information* after sufficient vertex misalignment

Thank you!

Simulated swarm data [HDSS24]

Video:

https://www.cis.jhu.edu/~parky//SofA/NRL-swarm-movie.mp4

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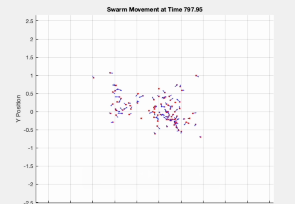
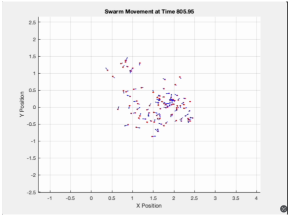
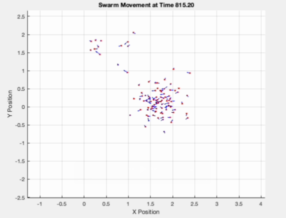


797s



805s



815s

We decide by eye there are two changepoints around 800s and 810s.

Simulated swarm data [HDSS24]

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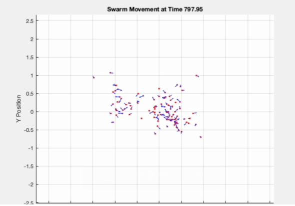

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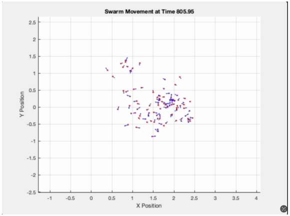

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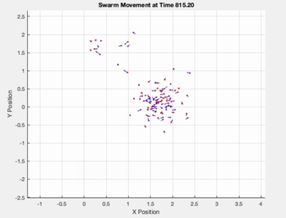


815s

We decide by eye there are two changepoints around 800s and 810s.

How do quantitatively find the changepoint for the dynamics of **overall** swarms?

Video to Time series of graphs

There are $m = 400$ frames, and in each frame there are $n = 100$ objects with locations in $\mathbb{R}^2$.

For each frame at time t,

1, *latent position matrix* $\mathbf{X}_t \in \mathbb{R}^{100\times 2}$ by stacking the locations row-wise,

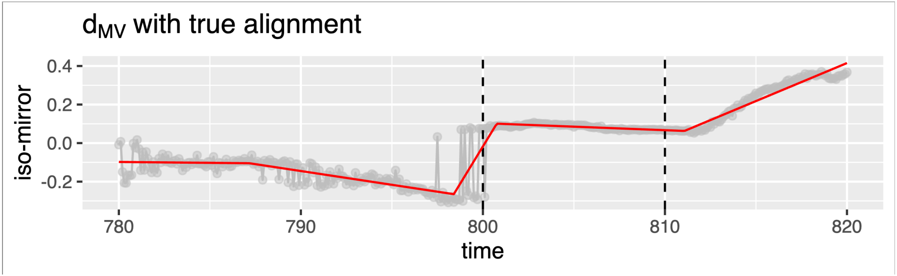
2, $\mathbf{P}_t = \mathbf{X}_t \mathbf{X}_t^\top \in \mathbb{R}^{100\times 100}$,

3, generate a graph using $\mathbf{P}_t$, $\mathbf{A}_{i,j} \sim \text{Bernoulli}(\mathbf{P}_{i,j})$.

Video to Time series of graphs

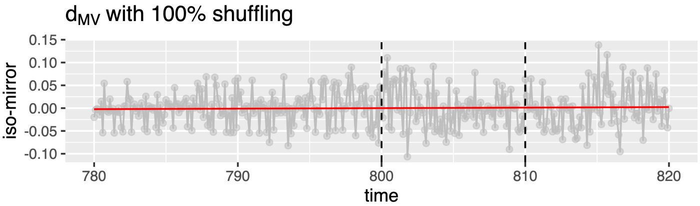
There are $m = 400$ frames, and in each frame there are $n = 100$ objects with locations in $\mathbb{R}^2$.
For each frame at time t,
1, *latent position matrix* $\mathbf{X}_t \in \mathbb{R}^{100\times 2}$ by stacking the locations row-wise,
2, $\mathbf{P}_t = \mathbf{X}_t \mathbf{X}_t^\top \in \mathbb{R}^{100\times 100}$,
3, generate a graph using $\mathbf{P}_t$, $\mathbf{A}_{i,j} \sim \text{Bernoulli}(\mathbf{P}_{i,j})$.
Given a latent position matrix and do step 2 and 3 is so called generating random dot product graph(RDPG) from a latent position matrix.

Euclidean mirror for simulated swarm data



Gray dots are iso-mirror of simulated swarm data, every dot represents a graph. Red line is the piecewise linear fit of the gray dots with number of slope changes chosen by BIC . We see slope change around 800 and 810.

What if there is vertex misalignment?



Gray dots are iso-mirror of 100% shuffled simulated swarm time series of graphs, every dot represents a graph. Signal are completely washed out.

References I

Avanti Athreya, Zachary Lubberts, Youngser Park, and Carey Priebe, *Euclidean mirrors and dynamics in network time series*, Journal of the American Statistical Association (2025), 1–12.

Jason Hindes, Kevin Daley, George Stantchev, and Ira B Schwartz, *Swarming network inference with importance clustering of relative interactions*, Journal of Physics: Complexity **5** (2024), no. 4, 045009.

d_{MV}, shuffled-d_{MV} for both models

Theorem 4. *For London model, with $\delta_m = \frac{1}{m}$, we have:*

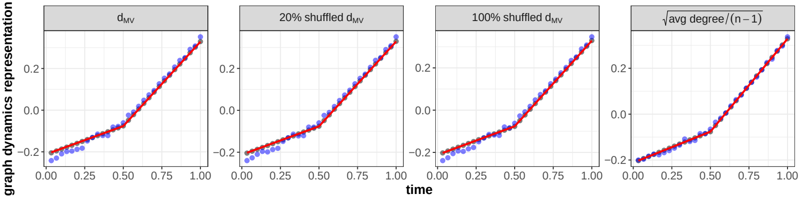
- *for d_{MV} distance:*
  - $$\left(\mathcal{D}^{\mathcal{L}}{}^{(2)}_{d_{MV}}\right)_{i,j} = d^2_{MV}(X^L_{t_i}, X^L_{t_j}) = \begin{cases} p^2\left(\frac{i}{m}-\frac{j}{m}\right)^2 + \frac{p-p^2}{m}\left|\frac{i}{m}-\frac{j}{m}\right| & i,j \le t^*_m, \\ \left(p(\frac{t^*_m}{m}-\frac{i}{m}) + q(\frac{j}{m}-\frac{t^*_m}{m})\right)^2 + \frac{p-p^2}{m}\left|\frac{t^*_m}{m}-\frac{i}{m}\right| + \frac{q-q^2}{m}\left|\frac{j}{m}-\frac{t^*_m}{m}\right| & i < t^*_m < j, \\ q^2\left(\frac{i}{m}-\frac{j}{m}\right)^2 + \frac{q-q^2}{m}\left|\frac{i}{m}-\frac{j}{m}\right| & t^*_m < i,j. \end{cases}$$
  - *as $m \to \infty$, $\tilde{\psi}^L_{d_{MV}}$ is asymptotically Euclidean 1-realizable with asymptotic mirror ψ_Z, that is,*
    $$\sup_{t\in[0,1]}\left|\tilde{\psi}^L_{d_{MV}}(t) - \psi_Z(t)\right| \to 0, \quad \left(\mathcal{D}^{\mathcal{L}}{}_{d_{MV}}\right)_{i,j} \to |\psi_Z(t_i) - \psi_Z(t_j)| \text{ as } m \to \infty \quad \text{for all } i,j \in \{1,2,\cdots,m\};$$
- *for shuffled-d_{MV} distance:*
  - $$\left(\mathcal{D}^{\mathcal{L}}{}^{(2)}_{shuffled\text{-}d_{MV}}\right)_{i,j} = shuffle\text{-}d^2_{MV}(X^L_{t_i}, X^L_{t_j}) = \begin{cases} p^2\left(\frac{i}{m}-\frac{j}{m}\right)^2 + \frac{p-p^2}{m}\left(\frac{i}{m}+\frac{j}{m}\right) & i,j \le t^*_m, \\ \left(p(\frac{t^*_m}{m}-\frac{i}{m}) + q(\frac{j}{m}-\frac{t^*_m}{m})\right)^2 + \frac{p-p^2}{m}\left|\frac{t^*_m}{m}+\frac{i}{m}\right| + \frac{q-q^2}{m}\left|\frac{j}{m}-\frac{t^*_m}{m}\right| & i < t^*_m < j \\ q^2\left(\frac{i}{m}-\frac{j}{m}\right)^2 + \frac{q-q^2}{m}\left|\frac{i}{m}+\frac{j}{m}\right| + \frac{2t^*_m}{m^2}\left(p-p^2-q+q^2\right) & t^*_m < i,j, \\ 0 & i=j. \end{cases}$$
  - *as $m \to \infty$, $\tilde{\psi}^L_{shuffled-d_{MV}}$ is asymptotically Euclidean 1-realizable with asymptotic mirror ψ_Z, that is:*
    $$\sup_{t\in[0,1]}\left|\tilde{\psi}^1_{shuffled\text{-}d_{MV}}(t) - \psi_Z(t)\right|, \quad \left(\mathcal{D}^{\mathcal{L}}{}_{shuffled\text{-}d_{MV}}\right)_{i,j} \to |\psi_Z(t_i) - \psi_Z(t_j)| \text{ as } m \to \infty \quad \text{for all } i,j \in \{1,2,\cdots,m\};$$
- *for α-shuffled-d_{MV} distance with $0 \le \alpha < 1$:*
  - $$\mathcal{D}^{\mathcal{L}}{}^{(2)}_{\alpha-shuffled\text{-}d_{MV}} = \alpha\mathcal{D}^{\mathcal{L}}{}^{(2)}_{shuffled\text{-}d_{MV}} + (1-\alpha)\mathcal{D}^{\mathcal{L}}{}^{(2)}_{d_{MV}},$$
  - *as $m \to \infty$, $\tilde{\psi}^L_{\alpha-shuffled-d_{MV}}$ is asymptotically Euclidean 1-realizable with asymptotic mirror ψ_Z, that is:*
    $$\sup_{t\in[0,1]}\left|\tilde{\psi}^L_{\alpha-shuffled-d_{MV}}(t) - \psi_Z(t)\right|, \quad \left(\mathcal{D}^{\mathcal{L}}{}_{\alpha-shuffled\text{-}d_{MV}}\right)_{i,j} \to |\psi_Z(t_i) - \psi_Z(t_j)| \text{ as } m \to \infty \quad \text{for all } i,j \in \{1,2,\cdots,m\};$$
- *for expected average degree, when $\frac{t^*_m}{m} = t^*$, for all $i \in [m]$:*
  $$\sqrt{\frac{\mathbb{E}\psi^L_{avg\text{-}degree}(t_i)}{n-1}} = \psi_Z(t_i).$$

Theorem 5. *For Atlanta model with N, m, c_A, t^*, we have:*

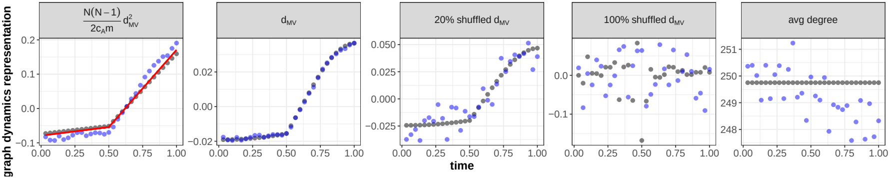
- *For the d_{MV} distance:*
  - $$\left(\mathcal{D}^{\mathcal{A}}{}^{(2)}_{d_{MV}}\right)_{i,j} = d^2_{MV}(X^A_{t_i}, X^A_{t_j}) = \begin{cases} \frac{c^2}{(N-1)^2N}\operatorname{tr}\left(T_p^{|i-j|}M\right) & i,j < t^*_m, \\ \frac{c^2}{(N-1)^2N}\operatorname{tr}\left(T_p^{|i-t^*_m|}T_q^{|t^*_m-j|}M\right) & i < t^*_m < j, \\ \frac{c^2}{(N-1)^2N}\operatorname{tr}\left(T_q^{|i-j|}M\right) & t^*_m < i,j, \end{cases}$$
    where $(M)_{i,j} = (i-j)^2$. Note for non-negative integer $2 \le k < N$, we have
    $$\operatorname{tr}\left(T_p^k M\right) = 2(N-1)kp - 4\sum_{t=2}^{k}\frac{(-1)^t}{t-1}\binom{k}{t}\binom{2t-4}{t-2}p^t, \tag{3}$$
    $$\operatorname{tr}\left(T_p^k T_q^l M\right) = 2(N-1)(kp+lq) - \sum_{d=2}^{k+l}\sum_{t=\max(0,d-l)}^{\min(d,k)}(-1)^d\frac{4}{d-1}\binom{2d-4}{d-2}\binom{k}{t}\binom{l}{d-t}p^t q^{d-t}. \tag{4}$$
  - *Further for fixed m as $N \to \infty$, and when $t^*_m = t^*m$, the first dimension of CMDS on $\mathcal{D}^{\mathcal{A}}{}_{d^2_{MV}}$ has property:*
    $$\max_{i\in[m]}\left|\frac{N(N-1)}{2c_A^2 m}\psi^A_{d^2_{MV}}(t_i) - \psi_Z(t_i)\right| \to 0, \quad \left|\frac{N(N-1)}{2c_A^2 m}\left(\mathcal{D}^{\mathcal{A}}{}_{d^2_{MV}}\right)_{i,j}\right| \to |\psi_Z(t_i) - \psi_Z(t_j)|.$$
- *For the 100%-shuffled-d_{MV} distance:*
  - $$\left(\mathcal{D}^{\mathcal{A}}{}^{(2)}_{shuffled\text{-}d_{MV}}\right)_{ij} = shuffled\text{-}d^2_{MV}(X^A_{t_i}, X^A_{t_j}) = \begin{cases} 2\operatorname{Var}(u) = \frac{c_A^2}{6}\frac{N+1}{N-1}, & i \ne j, \\ 0 & i = j. \end{cases}$$
  - *further the first $m-1$ dimensions of CMDS of $\mathcal{D}^{\mathcal{A}}{}_{shuffled\text{-}d_{MV}}$ is*
    $$\mathbf{\Psi}^{A\ 1:(m-1)}_{shuffled-d_{MV}} = \sqrt{\frac{c_A(N+1)}{6(N-1)}}\mathbf{U}_{m-1} \text{ where } \mathbf{U}_{m-1} \text{ is any matrix so that } \mathbf{U}_{m-1}^\top\mathbf{U}_{m-1} = I_{m-1}.$$
- *For α-shuffled-d_{MV} distance with $0 \le \alpha < 1$:*
  - $$\mathcal{D}^{\mathcal{A}}{}^{(2)}_{\alpha-shuffled\text{-}d_{MV}} = \alpha\mathcal{D}^{\mathcal{A}}{}^{(2)}_{shuffled\text{-}d_{MV}} + (1-\alpha)\mathcal{D}^{\mathcal{A}}{}^{(2)}_{d_{MV}}$$
  - *the first dimension of CMDS on $\mathcal{D}^{\mathcal{A}}{}_{\alpha-shuffled\text{-}d_{MV}}$ satisfy that*
    $$\psi^A_{\alpha-shuffled\text{-}d_{MV}} = \sqrt{(1-\alpha) + \frac{\alpha c_A(N+1)}{12(N-1)\lambda_1}}\psi^1_{d_{MV}}, \quad \text{where } \lambda_1 := \lambda_1\left(-\frac{1}{2}H\mathcal{D}^{\mathcal{A}}{}^{(2)}_{d_{MV}}H\right).$$
- *For the expected average degree,*
  $$\mathbb{E}[\psi^A_{avg\text{-}degree}(t_i)] = (n-1)\frac{c_A^2}{4} \quad \forall i \in [m].$$

You don't have to read this because i have figures for you!

Mirror and shuffled mirror for both models



London model: $n = 100$, $p = 0.3$, $q = 0.9$,$m = 30$, $t^* = 0.5$, $c_L = 0.1$, $\delta_m = 0.9/30$.



Atlanta model: $n = 1000$, $p = 0.05$, $q = 0.45$, $m = 30$, $t^* = 0.5$, $c_A = 0.8$, $N = 50$.