Filtered Kernel Density Estimation

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Summary

A modification of the kernel estimator for density estimation is proposed which allows the incorporation of local information about the smoothness of the density. The estimator uses a small set of bandwidths rather than a single global one as in the standard kernel estimator. It uses a set of filtering functions which determine the extent of influence of the individual bandwidths. Various versions of the idea are discussed. The estimator is shown to be consistent and is illustrated by comparison to the single bandwidth kernel estimator for the case in which the filter functions are derived from finite mixture models.

Keywords: Kernel Estimator, Multiple Bandwidths, Mixture Models, Density Estimation.

1. INTRODUCTION

The kernel density estimator has been studied widely since its introduction in Rosenblatt 1956 and Parzen 1962. Given i.i.d. data $x_1,...,x_n$ drawn from the unknown density α, the standard kernel estimator (SKE) is the single bandwidth estimator:

$$\hat{\alpha}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right). \tag{1}$$

See the recent books by Silverman 1986, Scott 1992 and Wand and Jones 1995 and the bibliographies contained therein, for a good introduction to kernel estimators. Much work has been done on selecting the optimal bandwidth h under different assumptions on α or different optimality criteria.

Alternatively, variable bandwidth kernel estimators are of the form

$$\hat{\alpha}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h_i} K\left(\frac{x - x_i}{h_i}\right), \tag{2}$$

or variations on this theme. Breiman et. al. 1977, and Abramson, 1982 are early papers in this field, while Terrel and Scott 1992 give a good discussion of the issues of this approach. These estimators require a choice of many bandwidths, and several approaches have been investigated. The obvious problem which may arise in these variable bandwidth estimators is that it is not always clear how to best incorporate *a priori* information about the local smoothness of the density into these estimators. Furthermore, these estimators usually break down in the tails where the data is sparse, and hence it is difficult to get good estimates of appropriate local bandwidths.

We propose a modification to the standard kernel estimator (1), first introduced in Rogers, Priebe, and Solka 1993, which uses a small number of bandwidths instead of either extreme exemplified by equations (1) and (2).

Figure 1(a) is an example of the kind of density this approach is meant to address. The two modes of this density obviously require different bandwidths. A single bandwidth kernel estimator must make a trade off between undersmoothing one mode or oversmoothing the other.

Thus, we wish to have a small number of bandwidths where each bandwidth is associated with a region of the support of the density. To this end we use a set of functions which “filter” the data. Basically, the filter will define the extent to which each local bandwidth is to be used for any particular data point. We can then construct a kernel estimator which is a combination of the kernel estimators constructed using each bandwidth, with the data filtered by the filtering functions. To be specific, consider a set of functions $\{\rho_j\}_{j=1}^{m}$ where $0 \le \rho_j(x) \le 1$ and

$$\sum_{j=1}^{m} \rho_j(x) = 1 \tag{3}$$

for all x. The ρ functions can be interpreted as probabilities and are used to incorporate prior information concerning local smoothness. We will refer to the ρ functions as filtering functions. Associate to each filtering function ρ_j a bandwidth h_j such that

$$\begin{aligned} 0 &< h_j \\ h_j &\to 0 \\ nh_j &\to \infty \end{aligned} \tag{4}$$

as $n \to \infty$. The filtered kernel estimator (FKE) for the filter $\{\rho_j\}_{j=1}^{m}$ is

$$\hat{\alpha}(x) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\rho_j(x_i)}{h_j}K\left(\frac{x-x_i}{h_j}\right). \tag{5}$$

The filtered kernel estimator was first formulated in terms of a mixture model: given a finite mixture

$$f(x) = \sum_{j=1}^{m}\pi_j f_j(x) \tag{6}$$

the filtering functions defined by the mixture are:

$$\rho_j(x) = \frac{\pi_j f_j(x)}{f(x)}. \tag{7}$$

The idea is to estimate the density as a finite mixture of the form (6) and use a bandwidth for each component which is in some sense optimal for that component under the overall mixture model and thus vary the bandwidth according to the individual variances of the filtering mixture. In practice, one would fit a mixture to the data which one felt was a good representative of the local variance of the underlying distribution, then use the mixture to construct bandwidths and a filtered kernel estimator. This approach works well even when the data is not distributed as a finite mixture, provided that the mixture captures enough of the local variance characteristics of the data.

Figure 1(b) shows the filter functions associated with the density in Figure 1(a). For illustration, we have used the true mixture to construct the filter functions. The effective kernels (the inner sum in (5)) associated with some points are shown overlaid in Figure 1(a). The bandwidths used in this example are 0.32 and 2.1, which are appropriate for n=1000. Note that different regions have different associated bandwidths, and that regions of overlapping filters have effective kernels which are mixtures of the individual kernels. This can be seen in the third component from the left, which has larger than normal tails.

The filtered kernel estimator does not require a finite mixture for the construction of the filtering functions, as will be seen below. Essentially any functions satisfying the conditions above can be used. The purpose of the filtering functions is to allow the user to specify the regions in which different bandwidths will operate.

We will use mixtures of normals throughout this paper for the construction of the filtering functions for two reasons: we have some experience in finite mixture estimation, and it is a convenient arena in which to do some comparisons with the standard kernel estimator. There are a number of methods for choosing the number of components to be used, either subjectively or using automated methods such as in Priebe 1994, Solka et al 1995 or Rogers et al 1995. In this paper, we will either assume knowledge of the true mixture (for comparison with the standard kernel estimator) or use subjective methods.

This idea of filtering the data is similar to that used in Wand, Marron and Rupert 1991, where the data is transformed to a known density (for example a normal) and the kernel estimator is performed there. One can think of the filters as pulling out the data from each component and then performing a kernel estimator on each component's data. The advantage the filtered kernel estimator has over the transformation approach (in our opinion) is that the relatively small number of bandwidths give easy control over the local smoothness of the estimator, something which must be built into the transformation in the Wand, Marron and Rupert estimator.

Note that the variable kernel estimator is a version of filtered kernel estimator: let m=n and $\{A_i\}$ be a disjoint partition of the support of the density such that A_i intersects the data in $\{x_i\}$ and let

$$\rho_j(x_i) = \chi_{A_j}(x_i) \tag{8}$$

Then it is easy to see that (5) reduces to (2). The philosophy of the filtered kernel estimator, however, is to use fewer bandwidths, and to use the information from the estimating mixture or *a priori* information to choose these bandwidths.

To better illustrate this philosophy, we make the following modification of the filtered kernel estimator (5) into an estimator with only a single bandwidth: let

$$\hat{\alpha}(x) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\rho_j(x_i)}{h\sigma_j}K\left(\frac{x-x_i}{h\sigma_j}\right), \tag{9}$$

where the σ_j are the standard deviations of the components of (6). Thus we are letting $h_j = h\,\sigma_j$ for a single bandwidth h. In effect what (9) does is place kernels with different variances at different points, the variances of the kernels being tied to the filtering mixture, and their contribution to the estimator also determined by the filtering mixture. While this is slightly less general than (5) it seems to work quite well in practice and the asymptotically optimal h can be computed in closed form, as will be seen below.

Recent work by Hjort and Glad 1995 has considered using a parametric density as a start on the nonparametric estimator. Given a parametric estimate $f(x, \hat{\theta})$ the authors define a new nonparametric estimator

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n}\frac{f(x, \hat{\theta})}{f\left(x_i, \hat{\theta}\right)}K\left(\frac{x_i - x}{h}\right). \tag{10}$$

If we assume the parametric estimator is a mixture of the form (6), we can rewrite this as

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\pi_j f_j(x)}{f(x_i)}K\left(\frac{x_i - x}{h}\right). \tag{11}$$

This is very close to the filtered kernel estimator, and it is clear that one can incorporate multiple bandwidths into this estimator in the same way. As the authors point out, this works well even if the parametric estimator is quite crude, which also holds for the filtered kernel estimator.

Although we are concerned here with univariate densities, the filtered kernel estimator has an interesting extension to multivariate densities. Assume that the kernel is a normal density and that the mixture (6) is a mixture of multivariate normals. For each local bandwidth h_j, we associate both the posterior probabilities from the mixture (the filtering function) and the covariance of the j^{th} component Σ_j. Thus we can take into account local structure as represented by the mixture approximation to the density. Thus, we have the estimator:

$$\hat{\alpha}(x) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho_j(x_i) \quad \varphi\left(x, x_i, h_j^2\Sigma_j\right) \tag{12}$$

or, in the case of (11) above,

$$\hat{\alpha}(x) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho_j(x_i) \quad \varphi\left(x, x_i, h^2\Sigma_j\right) . \tag{13}$$

Clearly, this can be extended to more general kernels. This allows the use of elliptical kernels which are better shaped to the local density, with different kernel shapes in different regions of the support.

2. ASYMPTOTICS

Assume the conditions on the h_j's in eqn (4). Assume further that K(t) is bounded, and a p^{th} order kernel, that is,

$$\int K(t)\,dt = 1 \qquad \lim_{|t| \to 0} |tK(t)| = 0$$

$$\int |K(t)|\,dt < \infty \qquad \int K^2(t)\,dt < \infty \tag{14}$$

$$\int t^r K(t)\,dt = 0 \qquad \int t^r K(t)\,dt = \pm 1$$

Theorem 1: Under the above conditions, and assuming the existence of p^{th} derivative of $\alpha\rho_j$, and that this derivative is in L_1, the filtered kernel estimator $\hat{\alpha}(x)$ is L_2 consistent.

pf: Recall that the mean integrated squared error (MISE) can be written as

$$MISE(\hat{\alpha}) = \int bias^2(\hat{\alpha}) + Var(\hat{\alpha})\,. \tag{15}$$

So the asymptotic bias is

$$\frac{1}{p!}\sum_{j=1}^{m} h_j^p \frac{d^p}{dx^p}(\alpha(x)\rho_j(x)) \tag{16}$$

and so the first term in (15) is

$$\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{h_j^p h_k^p}{(p!)^2} \int \frac{d^p}{dx^p}(\alpha(x)\rho_j(x)) \frac{d^p}{dx^p}(\alpha(x)\rho_k(x))\,dx\,. \tag{17}$$

Letting

$$g(h_j, h_k) = \int K\left(\frac{t}{h_j}\right)K\left(\frac{t}{h_k}\right)dt, \tag{18}$$

we have for the second term of (15)

$$\frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{g(h_j, h_k)}{h_j h_k}\int\rho_j(y)\rho_k(y)\alpha(y)dy \qquad . \tag{19}$$

Finally,

$$g(h_j, h_k) \le min(h_j, h_k)\, sup(K(t)) \ . \tag{20}$$

Combining equations (17) and (19) we have

$$AMISE = \left(\begin{array}{c}\displaystyle\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{h_j^p h_k^p}{(p!)^2}\int\frac{d^p}{dx^p}(\alpha(x)\rho_j(x))\frac{d^p}{dx^p}(\alpha(x)\rho_k(x))dx + \\ \displaystyle\frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{g(h_j, h_k)}{h_j h_k}\int\rho_j(y)\rho_k(y)\alpha(y)dy\end{array}\right. \ . \tag{21}$$

Thus, AMISE $\to 0$. Note that since the standard kernel estimator is a special (trivial) case of the filtered kernel estimator, this can always be made less than or equal to the AMISE for the SKE, and with appropriate choice of the filters and bandwidths can (in many cases) improve on the performance of the SKE.

If the kernel K is the standard normal we can compute g() and obtain

$$g(h_j, h_k) = \frac{1}{\sqrt{2\pi}}\frac{h_j h_k}{\sqrt{h_j^2 + h_k^2}}. \tag{22}$$

In keeping with the ideas discussed in the introduction, we assume in this section that α is a mixture of normals, and that the filtering functions are generated by the same mixture. Equation (21) then becomes

$$AMISE = \frac{1}{4}\sum_{j=1}^{m}\sum_{k=1}^{m} A_{jk} h_j^2 h_k^2 + \frac{1}{n\sqrt{2\pi}}\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{B_{jk}}{\sqrt{h_j^2 + h_k^2}} \tag{23}$$

where

$$A_{jk} = \pi_j \pi_k \int f_j''(x) f_k''(x)\, dx, \tag{24}$$

$$B_{jk} = \pi_j \pi_k \int \frac{f_j(x) f_k(x)}{\alpha(y)} dx. \tag{25}$$

In practice we first approximate the unknown density as a mixture, then minimize (23) (numerically) to calculate the bandwidths under the assumption that the filtering density is the true density. Thus we use the optimal values for h_j under the assumption that the filtering mixture is correct. This is analogous to using a reference density such as a normal to compute the bandwidth for the standard kernel estimator.

If instead of (5) we use (9), we have just a single bandwidth to choose, and, again assuming that $\alpha = f$, we have:

$$h_{opt} = \left(\frac{\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{B_{jk}}{\sqrt{\sigma_j^2 + \sigma_k^2}}}{\sqrt{2\pi}\; n \sum_{j=1}^{m}\sum_{k=1}^{m} A_{jk}\sigma_j^2\sigma_k^2} \right)^{\frac{1}{5}}, \tag{26}$$

which is quite similar to the formula for the standard kernel estimator, and in fact reduces to it when all the variances are equal. This can be seen by noting that

$$\sum_{j=1}^{m}\sum_{k=1}^{m} B_{jk} = 1$$

and

$$\sum_{j=1}^{m}\sum_{k=1}^{m} A_{jk} = \int (\alpha'')^2 .$$

3. EXAMPLES

We compare the AMISE of the FKE with the standard kernel estimator with h chosen optimally. When simulations are performed, the bandwidths are chosen by numerically minimizing (23) or by using (26). Following Wand, Marron, and Ruppert 1991, we compute the efficiency of the estimator as $\mathrm{AMISE}_{\mathrm{FKE}}/\mathrm{AMISE}_{\mathrm{SKE}}$ so small values of the efficiency correspond to better estimates with the FKE. For both the SKE and the FKE the true mixture is used to compute the optimal bandwidth(s). Thus, the filtering mixture for these examples is also taken to be the true mixture. This allows us to compare the two estimators in a "best case" scenario.

For the examples on data which is not from a normal mixture, the filtering mixture used is computed from a 2 component mixture via the EM algorithm. In both cases the number of components is chosen by noting that the data is clearly not distributed normally and that a 2 component mixture appears to be a reasonable (conservative) fit to the data.

Example 1: Let $\alpha(x) = \frac{1}{2}N(0, 1) + \frac{1}{2}N(0, v)$, with $0.1 \le \nu \le 10$.

Figure 2(a) shows the efficiency as a function of the log of the variance. Note that for $\nu \neq 1$, the FKE improves on the SKE, as one would expect. The bandwidths are shown in Figure 2(b), indicating that the FKE can use more appropriate bandwidths for these densities.

This is essentially the case that the FKE was designed to address. We have a density which is a mixture of two normals with unequal variances. As the variance of the second term is moved away from the variance of the first term, the standard kernel's single bandwidth becomes less and less appropriate for the resulting density. The filtered kernel estimator allows us to take the two variances into account in our estimator, thus improving the estimate when the variances are significantly different.

Example 2: Marron and Wand Densities.

Marron and Wand 1992 list 15 normal mixture densities showing some of the wide range of variations that are obtainable with simple mixtures. Table 1 shows the efficiency of the FKE for these densities. The SKE bandwidth is chosen to be optimal (asymptotically) under the mixture assumption. Note that the performance of the FKE depends on the amount of local variability of the mixture, as would be expected. The first efficiency column uses the bandwidths chosen by minimizing the $\mathrm{AMISE}_{\mathrm{FKE}}$ numerically, while the second column shows the efficiency making use of the variances of the components as in (9) and (26). Once again, in those instances where the use of multiple bandwidths is appropriate, the FKE shows improvement over the SKE. Note that it seems to make very little difference which method is used to choose the bandwidths for the FKE.

The above examples dealt with the theoretical properties of the FKE, where the filter is assumed to be equal to the underlying density. In practice this is not possible, and in fact if the underlying density is known any attempts at estimation are obviously unnecessary. In the next examples we consider the case where the underlying density is not known. In these cases we first fit a mixture to the data to obtain a reasonable filter. Then we compute the h_j under the assumption that the filter is equal to the density. In practice, as will be seen below, this provides a good estimator provided the filtering mixture captures most of the underlying variability of the data.

Example 3: Wiener index of hydrocarbons.

In chemical graph theory, one wants to characterize compounds by invariants calculated from the graphical representation of their molecular structure. The purpose is to use these invariants to infer properties (boiling point, mutagenicity, etc.) of new compounds. In this example we consider one such invariant, the Wiener index. Information on how this is calculated, and some discussion of the problem in general can be found in Basak et al 1995.

We have considered the Wiener index of 140 hydrocarbons, and have plotted two kernel estimators (solid lines) and a two component mixture fit to the data in Figure 3(a). Note that the data appears to be pretty well fit by the mixture model, however the mode (consisting of approximately 70% of the data) appears to be biased slightly to the left. Using this mixture model as our filter, we fit a filtered kernel estimator (bandwidths 12.5 and 140.9, from (26)) to the data, and this is presented in Figure 3(b) as the solid curve (we have focused on the main mode in this figure). The mixture model is represented as the dashed line. Note that the kernel estimator does appear to demonstrate a non-normal structure at the mode. Outside this range, the kernel estimator and the mixture agree pretty well, although the kernel estimator is slightly flatter then the mixture model. It is hard with so little data to decide between the two models on this long tail, and so we will concern ourselves with the mode in this example.

In order to illustrate the flexibility of the FKE we have used a smaller first bandwidth, in this case a bandwidth of 9 and plotted the result as the dotted line. Note that the effect of this reduced bandwidth is restricted to the region below 200. Thus, the user can adjust the smoothness of the estimator within regions indicated by the parametric approximation, without having a large effect on other regions of the density. It seems that the default bandwidths have done a pretty good job on this density. We would not argue that the smaller bandwidth is more appropriate in this case, but merely include it to illustrate the flexibility of the FKE.

Note that in spite of the figure, we are really interested in an estimate of the density throughout its range. Thus, one cannot simply truncate the data and estimate the density of the truncated data. The FKE gives us a function which is an estimate of the density, not just a sequence of plots which require the mental smoothing of some regions, which is what a standard kernel estimator is forced to do in many cases.

Example 4: Lognormal.

100 data points were drawn from a lognormal and a two component mixture was fit to the data using the EM method (see, e.g., Titterington, Smith, and Makov 1985). The bandwidths for the filtered kernel estimator were chosen assuming the filter to be equal to the true density. Thus we first construct the mixture estimate

and then use the bandwidths that would be optimal for that mixture density, in much the same way that one might use a normal or some other density as a reference estimate for the standard kernel estimator.

Figure 4(a) shows the density estimates for the standard kernel estimator and the FKE. The bandwidths for the FKE were $h_1 = 0.4$ and $h_2 = 2.2$. The bandwidth for the standard kernel estimator was chosen by hand to get a reasonable fit to the true density. The plot shown uses a value of 0.3 for the bandwidth of the standard kernel. Note that the FKE smooths the tail without over smoothing the mode. As in the Hydrocarbon data example we can decrease the first bandwidth to move the FKE closer to the SKE at the mode without effecting the smoothness of the tail, if this were desirable.

Figure 4(b) compares the density estimates of the FKE and the filtering mixture. The FKE is a better fit than the filtering mixture. It could be argued that this is in part due to the choice of filtering mixture, and that a mixture of 3 or 4 terms could do much better. This is true, provided the number of terms did not get too large (recall this data set contains only 100 points). However, the point is that the FKE has the ability to overcome the shortcomings of the filtering mixture, and hence acts as a hedge against misspecification of the mixture. In fact, it is possible to use the results of the FKE as feedback to the filtering mixture to increase the number of terms (see Rogers et al 1995 for some preliminary work in this area).

4. DISCUSSION

The filtered kernel estimator is superior in performance to the standard kernel estimator, provided appropriate filter functions and bandwidths can be chosen. In section 2 it was shown that any reasonable filter functions will give asymptotic performance no worse than the standard single bandwidth kernel estimator. This is no surprise, since the single bandwidth kernel estimator is a special case of the filtered kernel.

It would seem at first that the added trouble of selecting filtering functions and bandwidths would make the estimator difficult to use in practice. However the idea of using a finite mixture fit to the data to construct the filters is one which appears to work well in a variety of situations, even those for which the data is not drawn from a finite mixture. The ability to take local structure into account is a powerful one which will allow much better estimates in those situations where there is reason to believe the local structure is justified. Experience has shown that good results are obtained even with a very crude estimate from the mixture model.

We suggest that in using this approach one should start with a very conservative estimate of the number of components. This will tend to give good results, and will not have the problems associated with overfitting the mixture model. In an exploratory data analysis mode one could then increase the number of components of the mixture model if the FKE seemed to indicate this was appropriate.

It should be noted that bad filters can produce bad FKE's. This is not unreasonable, however it does mean that care must be used in the choice of the filtering mixture. Just as the standard kernel estimator produces errors when the bandwidth is taken to be too large or too small, mixtures which have terms which are not supported by the data will produce local errors in the FKE estimate. This local character of the estimator gives some protection, since the effect of the error is reduced outside the region in which the corresponding filter function dominates. This is in contrast to the single kernel estimator where the choice of the bandwidth has a global effect.

The possibility of using the filtered kernel estimator to aid in choosing a mixture estimate of the density, as mentioned above, is quite interesting. Work on automated procedures to do this is ongoing, and preliminary results are reported in Rogers et al 1995.

As was noted in the introduction, we do not view the filtered kernel estimator as a version of a variable bandwidth kernel estimator of the kind in (2). However, the basic philosophy of using the components of a mixture to determine the bandwidths could be adopted for variable kernel estimators. Consider instead of having a single bandwidth h_j for each component, having multiple bandwidths $h_{ij} = h_j/\mathrm{sqrt}(f_j(x_i))$. Note that we still only have m bandwidths, the h_j, to choose, the rest coming immediately from the filter and the data. It is not obvious that this is an improvement over the traditional variable bandwidth estimators, using the filtering mixture as the pilot estimate for example, except that it retains the ability to easily change the estimate locally without effecting other regions. In the case of the filtered kernel estimator this is done by changing just one of the h_j's, as was demonstrated in the hydrocarbon data example. One can do this with the standard version of the variable kernel estimator in principle, but with quite a bit more work.

We have used mixtures to construct the filtering functions, and have shown that this works well in practice. It would be interesting to see if one could craft the filtering functions by hand, perhaps with a graphical interface to draw the functions. This might be a useful tool for data analysis. It is not obvious that the best filters are obtained from mixtures, even in the case that the density is a mixture, and more work is needed in this area.

Finally, we have focused on the univariate case in this work, but other extensions are possible. The multivariate version of the FKE has much promise, and will be addressed in the future. The ability to effectively tune the kernels to the local structure of the data will be a powerful and useful tool for multivariate density estimation. It is believed that this ability to define the structure locally will be of use in exploratory data analysis and in discriminant analysis.

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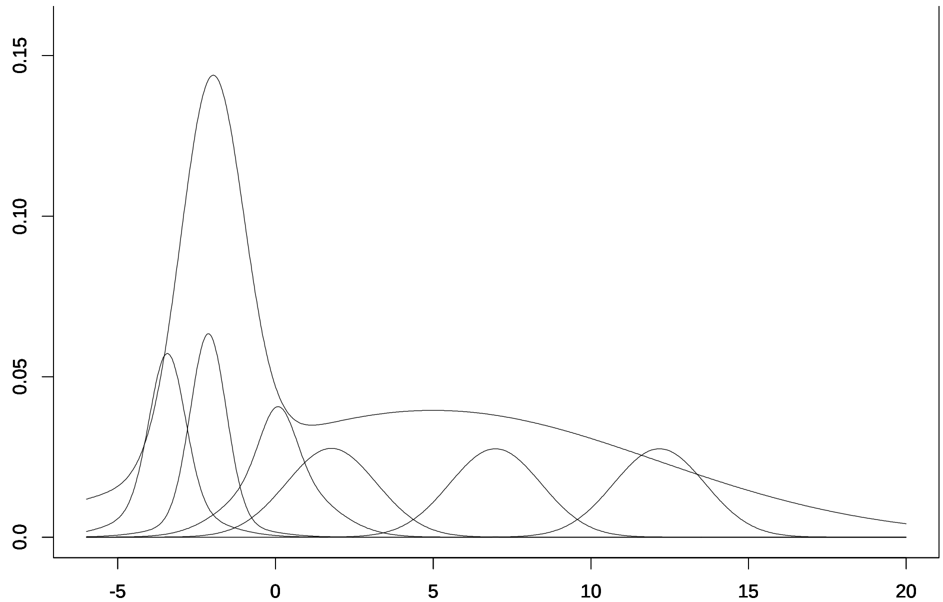
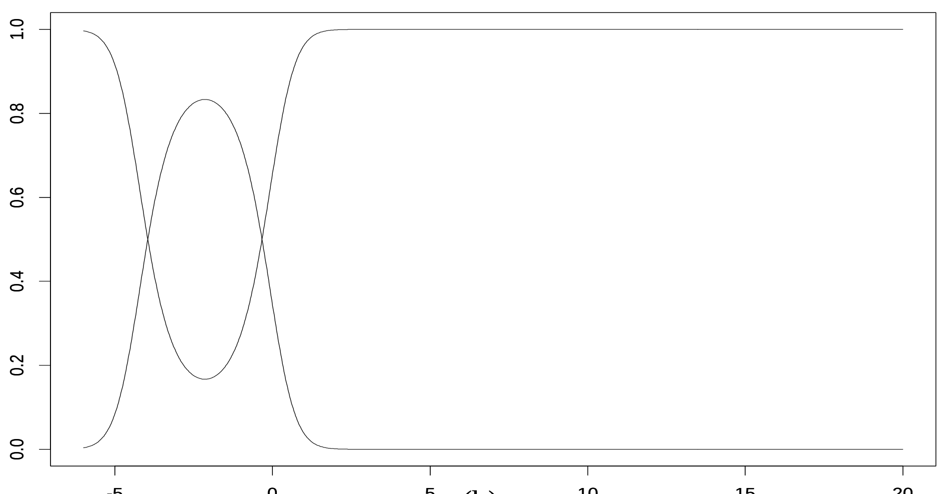
Table 1:

Density	Efficiency	Efficiency, $h_j = h\sigma_j$
Gaussian	1	1
Skewed Unimodal	.88	.9
Strongly Skewed	.38	.4
Kurtotic Unimodal	.44	.44
Outlier	.91	.91
Bimodal	1	1
Separated Bimodal	1	1
Skewed Bimodal	.69	.69
Trimodal	.90	.91
Claw	.51	.52
Double Claw	.13	.13
Asymmetric Claw	.52	.55
Asymmetric Double Claw	.15	.17
Smooth Comb	.33	.39
Discrete Comb	.5	.52



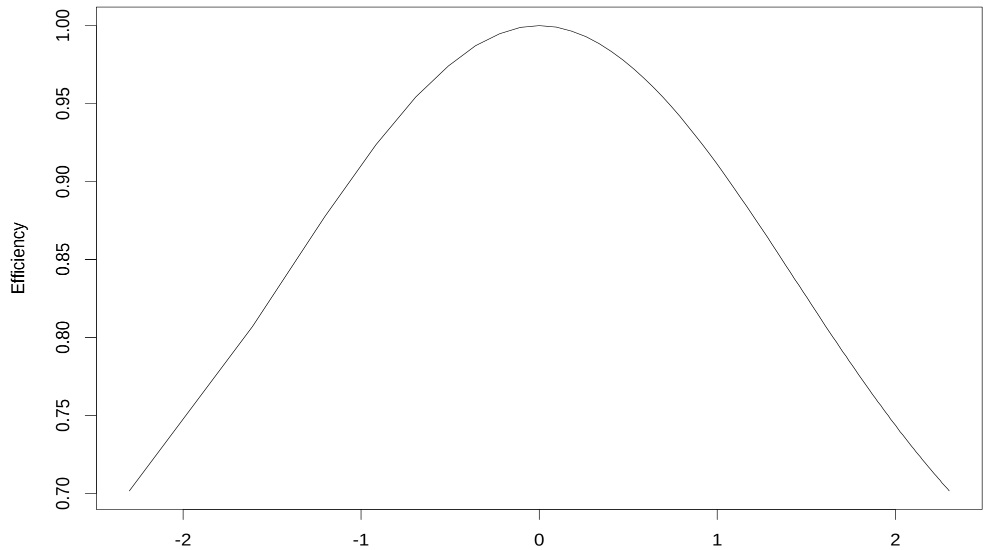
(a)



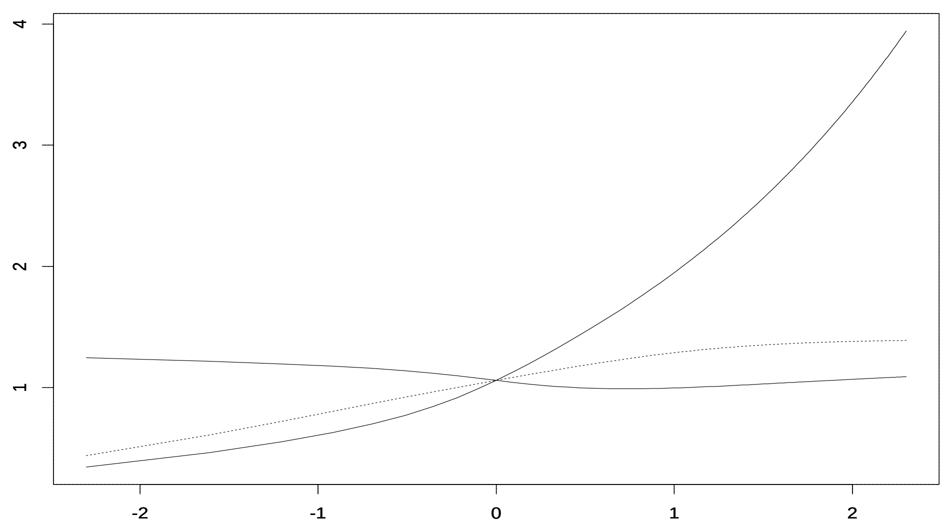
(b)

Figure 1

An example of the filtered kernel estimator applied to the two component mixture 0.3N(-2,1) + 0.7N(5,50). The mixture probability density function and ρ functions are shown. Example kernels are shown overlaid in (a).



(a)

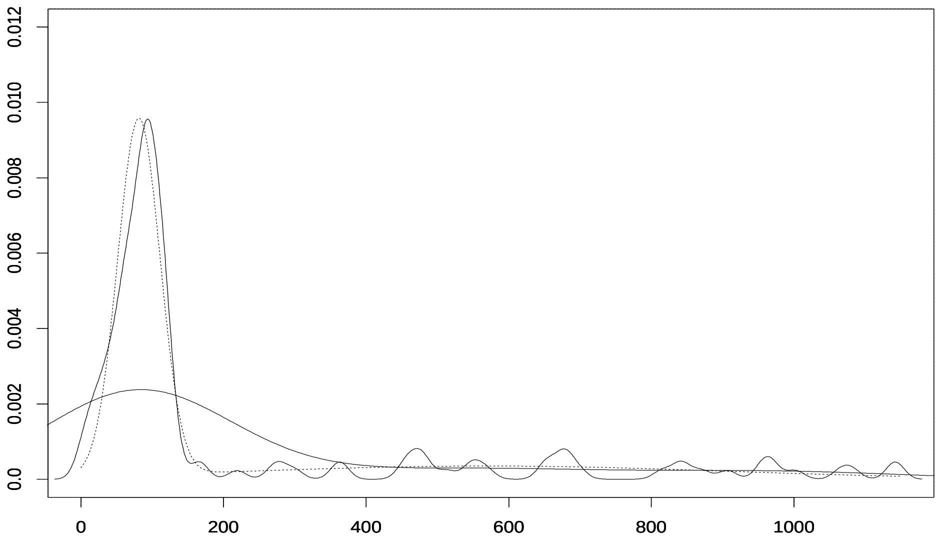


(b)

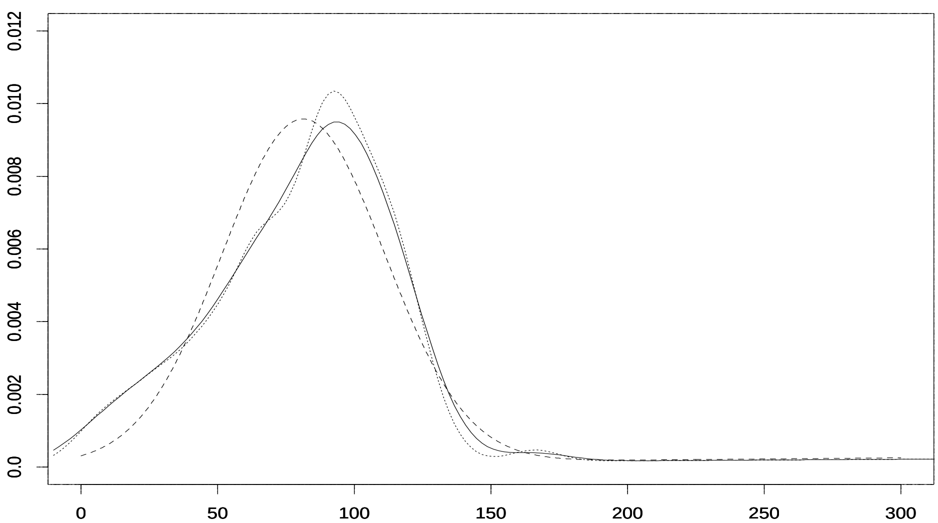
Figure 2

(a) Efficiency as a function of log(variance) for Example 1.

(b) The two bandwidths for the FKE are shown as solid lines, with the single bandwidth of the standard kernel estimator shown as a dotted line.



(a)



(b)

Figure 3

(a) Two kernel estimators (solid curves) and a 2 component mixture fit to the hydrocarbon Wiener Index data.

(b) The FKE with “optimal” bandwidths (solid curve) and user chosen bandwidths (dotted curve) and the mixture model (dashed curve) at the mode.

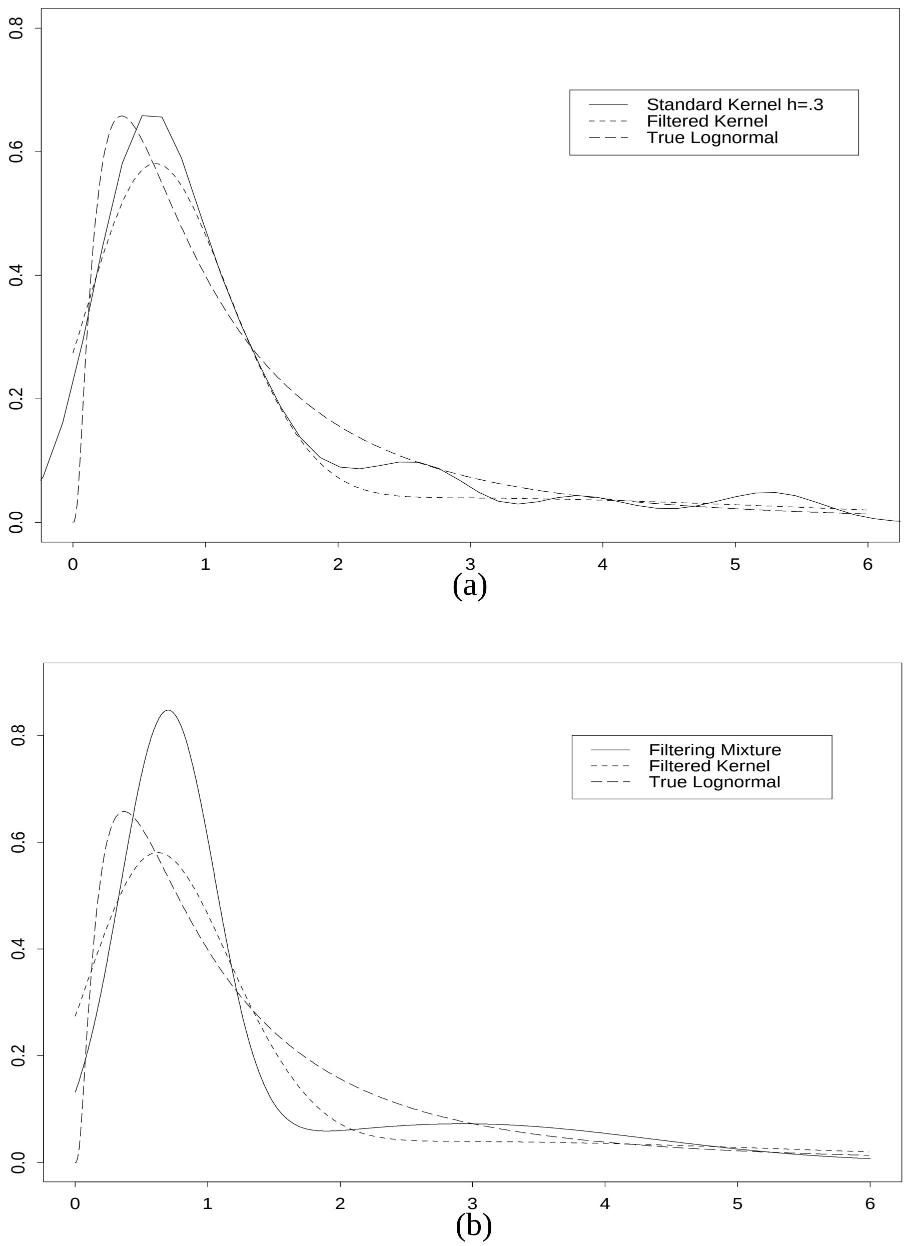


Figure 4

Density estimates for the standard kernel estimator, a mixture model and the FKE, along with the true lognormal density (Example 4).