Neighborhood Homogeneous Labelings of Graphs

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Abstract

Given a labeling of the vertices and edges of a graph, we define a type of homogeneity that requires that the neighborhood of every vertex contains the same number of each of the labels. This homogeneity constraint is a generalization of regularity – all such graphs are regular. We consider a specific condition in which both the edge and vertex label sets have two elements and every neighborhood contains two of each label. We show that vertex homogeneity implies edge homogeneity (so long as the number of edges in any neighborhood is four), and give two theorems describing how to build new homogeneous graphs (or multigraphs) from others.

Keywords: vertex labeling; edge labeling; homogenous graph; regular graph

1. Introduction

In [5] is considered the problem of finding an inhomogeneity in a time series of communication graphs. This is a problem in random graphs, where the inhomogeneity corresponds to a small number of vertices with a higher probability of connections within the group as otherwise. In [6, 4], this was extended to include a measure of the content of the communications – a label or color on each edge.

In this paper we consider a related question in non-random graphs. We consider graphs having labels on both the edges and the vertices, and ask questions about homogeneity in these graphs. Roughly speaking, we are interested in characterizing labeled graphs in which every neighborhood "looks the same"; in particular, the numbers of distinct vertex labels are always the same, as are the numbers of edge labels.

[3] provides a comprehensive survey of graph labeling which, as of the Nov. 2010 version, does not consider the types of labelings we are concerned with.

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For the most part, the labelings discussed in [3] use properties of the integers (the labels) to define various types of labelings. In our work, the labels are considered categorical, with no particular meaning. We will sometimes refer to the labels as colors, although this risks confusion with the topic of graph coloring, in which no adjacent vertices (or incident edges) are allowed to share a color.

1.1. Basic Notation

A graph is a pair $G = (V, E)$, where V is a finite set, the vertices, and E is a set of pairs of vertices, the edges. We will write vw for the edge $\{v, w\}$. We assume all graphs are simple and connected, unless stated otherwise. All edges are undirected. The number of vertices of a graph is called the order of the graph; the number of edges is called the size.

An induced subgraph of a set of vertices $W = \{w_1, \ldots, w_k\} \subset V$ is the graph $\Omega(W) = (W, E')$, where E' is the subset of E consisting of those pairs containing only elements of W: $E' = \{w_i w_j \in E | w_i, w_j \in W\}$. Thus, the induced subgraph contains all the edges between its vertices that exist in the original graph.

The open neighborhood of a vertex v, denoted $N(v)$, is the set of vertices $\{w \in V | vw \in E\}$; it consists of all neighbors of v. Note that $v \notin N(v)$. The closed neighborhood of v, denoted $N[v]$, is the set $N(v) \cup \{v\}$. We will denote by $G_v = \Omega(N[v])$ the induced subgraph of the closed neighborhood of v. Henceforth when referring to a neighborhood we will always mean the induced subgraph of the closed neighborhood. For graph theoretical terms not defined here, the reader is referred to [1].

A multigraph allows multiple edges between vertices. When we say that G is a graph, we mean that it is a simple graph with no loops or multiple edges. If we say that G is a multigraph we mean that it may have (but does not necessarily have) multiple edges, and all multigraphs will be loopless.

1.2. Neighborhood Labeling

Given two (disjoint, non-empty) sets $\mathcal{L}_v = \{\mathrm{vc}_1, \ldots, \mathrm{vc}_{k_v}\}$ and $\mathcal{L}_e = \{\mathrm{ec}_1, \ldots, \mathrm{ec}_{k_e}\}$ of labels, a vertex labeling is a map $l_v : V \to \mathcal{L}_v$ and an edge labeling is a map $l_e : E \to \mathcal{L}_e$. That is, a labeling assigns a label to each vertex/edge. When there is no possibility of confusion, we drop the subscript, and denote by $l(x)$ the label associated to the vertex or edge denoted by x. For notational convenience, we assume $\mathcal{L}_v \cap \mathcal{L}_e = \emptyset$.

A labeled graph is a graph G with vertex and edge labelings l_v, l_e: $G = (V, E, l_v, l_e)$. For any labeled graph G, define $n_i(G)$ to be the number of vertices (edges) of label $i \in \mathcal{L}_v$ ($i \in \mathcal{L}_e$).

A vertex labeling of G is an (a, b) homogeneous neighborhood vertex labeling (NVL) if $|\mathcal{L}_v| = a$ and $n_x(G_v) = b$ for all $v \in V$ and $x \in \mathcal{L}_v$ and say that G can be (a, b) neighborhood vertex labeled (both abbreviated (a, b)-NVL).

An edge labeling of G is a (c, d) homogeneous neighborhood edge labeling (NEL) if $|\mathcal{L}_e| = c$ and $n_x(G_v) = d$ for all $v \in V$ and $x \in \mathcal{L}_e$ and say that G can be (c, d) neighborhood edge labeled (both abbreviated (c, d)-NEL). Finally, a graph is $(a, b; c, d)$-NL (neighborhood labeled) if it is both (a, b)-NVL and (c, d)-NEL. Clearly in this case $ab - 1 \leq cd$, since the number of edges in the neighborhood is at least one less than the number of vertices. Note that we drop the word "homogeneous" throughout in the notation; it should always be understood that homogeneity (equal numbers of each label) is assumed.

2. Properties of Neighborhood Labelings

While $(1, b)$-NVL and $(1, d)$-NEL put conditions on the graphs that are non-trivial – the first implies that the graph is $(b-1)$-regular and the second that each neighborhood contain exactly d edges – they are trivial labelings on the graph, and so will not be considered here.

$(a, 1)$-NVL graphs are related to fall colorable graphs (see [7]). A graph G is fall a-colorable if G has a proper coloring (no edge has same-colored incident vertices) and every closed neighborhood contains all a colors. Clearly, since every neighborhood in G, an $(a, 1)$-NVL graph, has all the colors (labels) exactly once, this is a proper coloring, and G has chromatic number $\chi(G) = a$. This is also related to rainbow vertex coloring ([2]).

It is easy to see that there is only one $(2, 1)$-NVL connected graph – K_2, with two vertices and one edge, and the $(2, 1)$-NEL connected graphs are all even cycles. Thus, there are no $(2, 1; 2, 1)$-NL graphs. More generally, we have the following Proposition.

Proposition 2.1. *Suppose G is a $(2, k; 2, s)$-NL graph. Then for any vertex $v \in G$:*

(i). G_v has $2k$ vertices and $2s$ edges. Also $s \geq k \geq 1$.

(ii). G is $(2k-1)$-regular.

(iii). G_v contains a 3-cycle and so G is not bipartite.

(iv). The order of G is even.

(v). If G is a complete graph, then $G \cong K_{4m}$ for some $m \geq 1$.

(vi). If G is connected, then G_v is not a clique unless G is a complete graph.

Proof. (*i*) and (*ii*) are obvious.

(*iii*) G_v is not a tree, since the number of edges $2s$ is greater or equal to the number of vertices $2k$. Thus, it contains a cycle, and since all vertices (except v) are adjacent to v, it contains a 3-cycle.

(*iv*) Let n be the order of G, and e the size of G. Then

$$e = \frac{n(2k-1)}{2},$$

and so n is even.

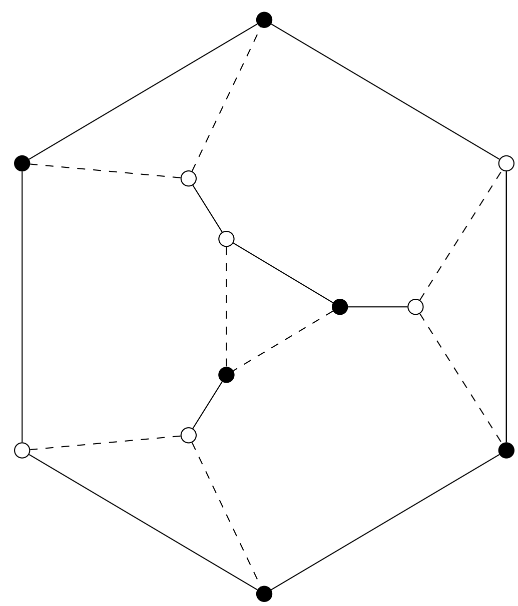


Figure 1: A $(2,2;2,2)$-NL graph. Vertex labels are indicated as solid/empty, and edge labels solid/dashed.

(v) Suppose G is a complete graph. Then $G_v \cong G$, and $2s = \frac{n(n-1)}{2}$. Since $n-1$ is odd, $n = 4m$ for some $m \geq 1$.

(vi) Suppose G is not complete but G_v is a clique for some $v \in V$. For any vertex $w \in V \setminus N[v]$, since G is connected, there is a path from w to some vertex in $N[v]$. Let z denote a vertex on this path that is distance one from $N[v]$, and $y \in N[v]$ a vertex connected to z. Then $z \in N[y]$ implying that $|N[y]| \geq 2k+1$ which is a contradiction. □

We now consider the simplest non-trivial case of neighborhood labelings, the $(2,2;2,2)$-NL graphs. Figure 1 depicts one such graph. The following follows immediately from Proposition 2.1.

Let G be $(2,2;2,2)$-NL. Then

1. G is 3-regular.

2. For each $v \in G$, G_v has 4 vertices, 4 edges, and is isomorphic to the graph in Figure 2 (i).

3. Referring to Figure 2 (i), the vertex c is on a 3-cycle in G_c (Figure 2, (ii)). Thus every edge is either on a 3-cycle, or connecting two 3-cycles. Call the former a t-edge and the latter a p-edge.

4. Since the degree of d in Figure (2)(ii) is 2, G has either an edge da (or isomorphically db) or a 3-cycle which is connected to d by a p-edge (Figure 3 (ii)).

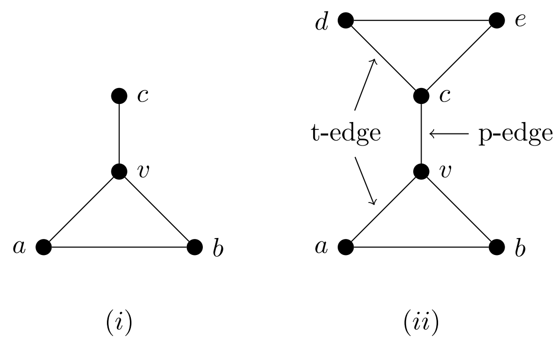


Figure 2: (i) The neighborhood of any vertex v in a $(2, 2; 2, 2)$-NL graph. (ii) Definition of p- and t-edges.

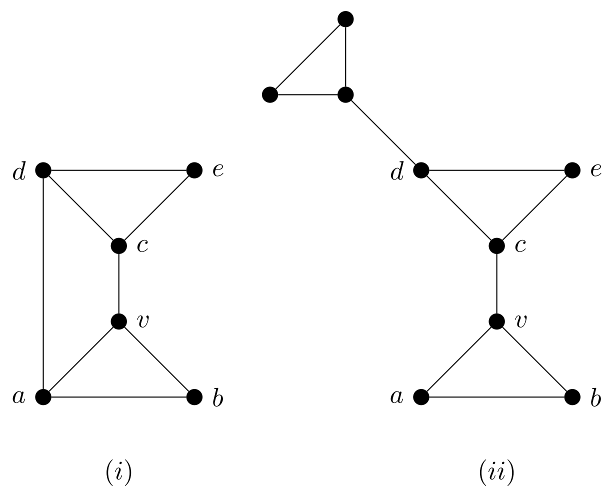


Figure 3: Two subgraphs used in the text in building up a $(2, 2; 2, 2)$-NL graph.

5. If we construct G from Figure 2 (ii), we are always adding either one p-edge or a p-edge and one 3-cycle. Thus the order of G is a multiple of 3 (Figure 3).

6. If two p-edges are on an ℓ-cycle with $\ell > 3$, they do not have a common end vertex. The same is true for t-edges. Hence, p-edges and t-edges alternate on such a cycle.

Proposition 2.2. *The order of a* $(2, 2; 2, 2)$*-NL graph is a multiple of* 6*.*

Proof. This follows immediately from Proposition 2.1 (iv) and 5 above. □

Proposition 2.3. *The length of a cycle in a* $(2, 2; 2, 2)$*-NL graph is either* 3 *or even.*

Proof. This follows immediately from 6 above. □

Proposition 2.4. *If each G_v in a $(2,2)$-NVL graph G has 4 edges, then G is $(2,2)$-NEL, and so is $(2,2;2,2)$-NL. Equivalently, every $(2,2)$-NVL graph that is not K_4 whose G_v contains 4 edges is $(2,2;2,2)$-NL.*

Proof. Since G is 3-regular, G_v for each $v \in G$ is isomorphic to the graph in Figure 2 (i). Let $\Sigma = \{1,2\}$ be a set of two (edge) labels. Assign label 1 to all p-edges. For each 3-cycle, label one edge with 1 and two edges with 2. This produces a $(2,2)$-NEL for G since an edge labeling scheme of a 3-cycle is independent from that of other 3-cycles. □

See Figure 1 for an example – note that the p-edges all have the same label (indicated by solid lines).

From a $(2,2;2,2)$-NL graph G, we can construct a loopless multigraph $\widetilde{G}$ by replacing each 3-cycle in G with a "thick" vertex. The resulting "collapsed" multigraph $\widetilde{G}$ is also 3-regular, with at most three edges between any pair of vertices. The definitions of NVL, NEL, and NL all extend naturally to multigraphs with the following important addition: vertices are counted according to the number of edges connecting them. If $\widetilde{w}$ and $\widetilde{x}$ are of distance one from $\widetilde{v}$ with (i) two edges between $\widetilde{w}$ and $\widetilde{v}$, and (ii) one edge between $\widetilde{x}$ and $\widetilde{v}$, we view the neighborhood of $\widetilde{v}$ as containing three vertices, two copies of $\widetilde{w}$ and one of $\widetilde{x}$. This duplication is used when counting the number of copies of each label in the neighborhood. See Figure 7 (ii) for an example of this.

The following theorem shows that we can determine whether a given 3-regular graph is $(2,2;2,2)$-NL by considering only smaller, collapsed versions of the graph. Further, we can build up larger graphs from smaller ones, by replacing vertices with 3-cycles, and retaining the $(2,2;2,2)$-NL property.

Theorem 2.1. *Collapsing Theorem.*

(i). If G is $(2,2;2,2)$-NL, the corresponding collapsed multigraph $\widetilde{G}$ is $(2,2)$-NVL.

(ii). If a loopless multigraph $\widetilde{G}$ is $(2,2)$-NVL, the graph G obtained from $\widetilde{G}$ by replacing each vertex with a 3-cycle, while each edge of $\widetilde{G}$ becomes a p-edge between two corresponding 3-cycles, is $(2,2;2,2)$-NL.

Proof. (i) Let $\Lambda = \{1,2\}$ be the set of vertex labels of the $(2,2;2,2)$-NL graph G. For a 3-cycle C_v of G, we may assume without loss of generality that the vertices of C_v are labeled with two 1s and one 2 (see Figure 4). Since G is 3-regular, each vertex on C_v is adjacent to a vertex not on C_v. If we name them x, y, and z as in Figure 4, they are all labeled with label 2 (white in the Figure).

There are 3-cycles C_x, C_y, C_z containing x, y, z respectively. Since G is $(2,2)$-NVL, the bottom two vertices on C_z in Figure 4 are both labeled 1; the two unlabeled vertices in each of C_x and C_y in Figure 4 are labeled with one 1 and

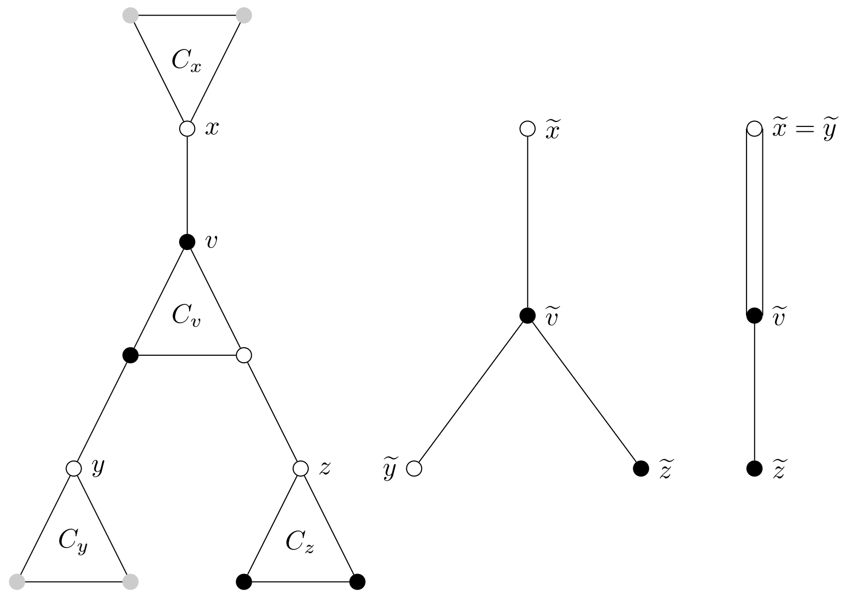


Figure 4: Local labeling around a cycle C_v in a $(2,2;2,2)$-NL graph G. Label 1 is black, label 2 is white, gray nodes are unspecified. The middle figure shows the collapsed version when the cycles C_x and C_y are distinct, the right-most Figure shows the result when they are the same.

one 2.[1] Notice that the 3-cycles C_x, C_y, C_z may not all be distinct. If they are not, we must have $C_x = C_y$, or x and y are on the same 3-cycle.

Now collapse the 3-cycles C_v, C_x, C_y, C_z into thick vertices $\widetilde{v}, \widetilde{x}, \widetilde{y}, \widetilde{z}$ and label these $1, 2, 2, 1$, respectively. The labeling scheme is to assign to the thick vertex the majority label from the cycle. This labeling scheme is then extended to all the vertices of G and thus $\widetilde{G}$ is $(2,2)$-NVL.

(ii) As above, let $\Lambda = \{1, 2\}$ be the set of vertex labels of a $(2,2)$-NVL of $\widetilde{G}$. For a vertex $\widetilde{v} \in \widetilde{G}$, let $\widetilde{x}$, $\widetilde{y}$ and $\widetilde{z}$ be vertices of distance 1 from $\widetilde{v}$ (note – these may not be distinct). We may assume without loss of generality that $\widetilde{v}$, $\widetilde{x}$, $\widetilde{y}$ and $\widetilde{z}$ are labeled 1, 2, 2, and 1 (thus, if these are not distinct, we must have $\widetilde{x} = \widetilde{y}$, since $\widetilde{G}$ is 3-regular; see Figure 5). When we expand the thick vertices to 3-cycles, we connect the 3-cycles so that each edge of $\widetilde{G}$ becomes a p-edge between two corresponding 3-cycles, and label the new vertices according to:

- If the end vertices of an edge in $\widetilde{G}$ have the same label, the end vertices of the p-edge connecting the corresponding 3-cycles in G get the same label.
- Otherwise the end vertices of the p-edge switch labels.

[1] These two pairs of gray vertices could be arbitraryily labeled so long as there one of each color in each cycle; we leave them gray to indicate that this labeling is arbitrary and not forced by the labeling of the middle graph.

See Figures 5 and 6 for illustration. This local vertex labeling scheme can be expanded to all vertices of G and so G is $(2,2)$-NVL. Since each subgraph of G induced by a closed neighborhood of a vertex has 4 edges, G is also $(2,2)$-NEL by Proposition 2.4.

□

By Theorem 2.1, a $(2,2;2,2)$-NL graph can be obtained from a $(2,2)$-NVL loopless multigraph. Figures 1 and 7 show the two $(2,2;2,2)$-NL graphs of order 12 (the 4-vertex thick multigraph that generates the graph in Figure 1 is the complete graph K_4). It is easy to see that K_4 and the bottom right multigraph in Figure 7 are the only 4-vertex $(2,2)$-NVL multigraphs. Clearly the smallest $(2,2)$-NVL multigraph has 4-vertices, and so we have:

Proposition 2.5. *The smallest $(2,2)$-NVL multigraph has 4 vertices, and there are exactly two of these, K_4 and M_4 – the one illustrated in Figure 7, and these result in the two $(2,2;2,2)$-NL graphs on 12 vertices depicted in Figures 1 and 7, bottom left. Thus there are exactly two $(2,2;2,2)$-NL graphs of order 12.*

Theorem 2.2. *If G is a $(2,2)$-NVL multigraph with $n > 4$ vertices, then there exists a (possibly disconnected) multigraph G^* which is $(2,2)$-NVL with $n-4$ vertices.*

Proof. Call the multigraph on the bottom right of Figure 7 M_4. If G is a $(2,2)$-NVL multigraph, then so is every connected component of G. Thus, if G contains K_4 or M_4 as a connected component, this can be removed, removing 4 vertices and leaving a $(2,2)$-NVL multigraph. So without loss of generality we may assume that G is connected.

The vertices we remove will always be the 4 vertices in the closed neighborhood of a vertex v, except in the case where there is a multiple edge (see Figures 9, 11–15). It is possible to connect the remaining vertices of G in such a way that the resulting multigraph G^* is $(2,2)$-NVL. The proof of this claim is relegated to the appendix. □

We can reduce any $(2,2)$-NVL multigraph through subtracting quadruples of vertices using Theorem 2.2 to either K_4 or M_4. Thus we can construct any graph by reversing this process – adding the vertices in the opposite order. Thus, any $(2,2)$-NVL multigraph has $4m$ vertices, for $m \geq 1$, and since by Proposition 2.2 the order of a $(2,2;2,2)$-NL graph is a multiple of 6, we have the following corollary:

Corollary 2.1. *The order of any $(2,2;2,2)$-NL graph is a multiple of 12.*

Note that since we can obtain either of these atomic multigraphs from the other, by first adding the other as a connected component then removing the first, we can obtain any $(2,2)$-NVL graph or multigraph by starting with K_4 and adding 4 vertices at a time (with at most one subtraction of 4 vertices) using the reverse of Theorem 2.2. We will say that a multigraph is *constructable* from a graph G if it can be obtained from G by a process of adding 4 vertices at a time

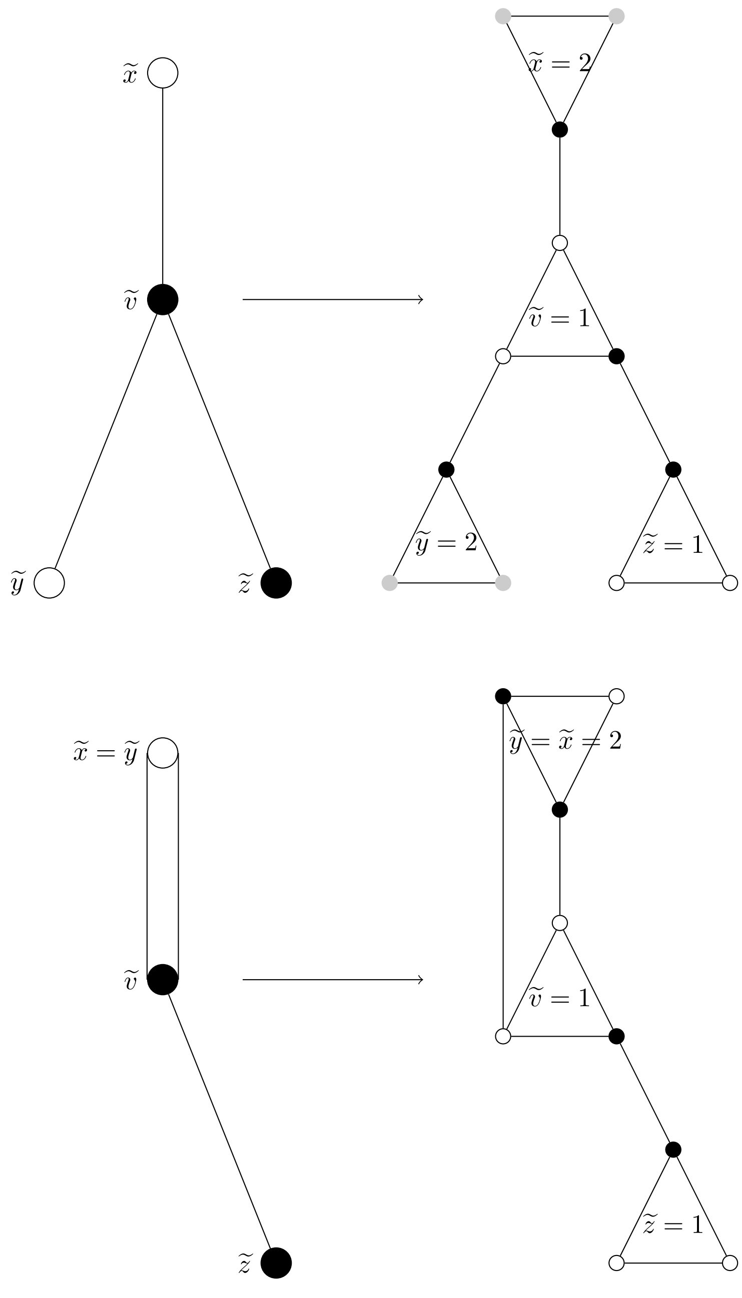


Figure 5: Expanding thick vertices to 3-cycles, illustrating the labeling scheme for the new vertices. Again label 1 is black, label 2 is white, gray nodes are unspecified, but in each 3-cycle, one gray node is black (1) and one is white (2). See Figure 6, which illustrates the effect of repeated expansions and collapsings.

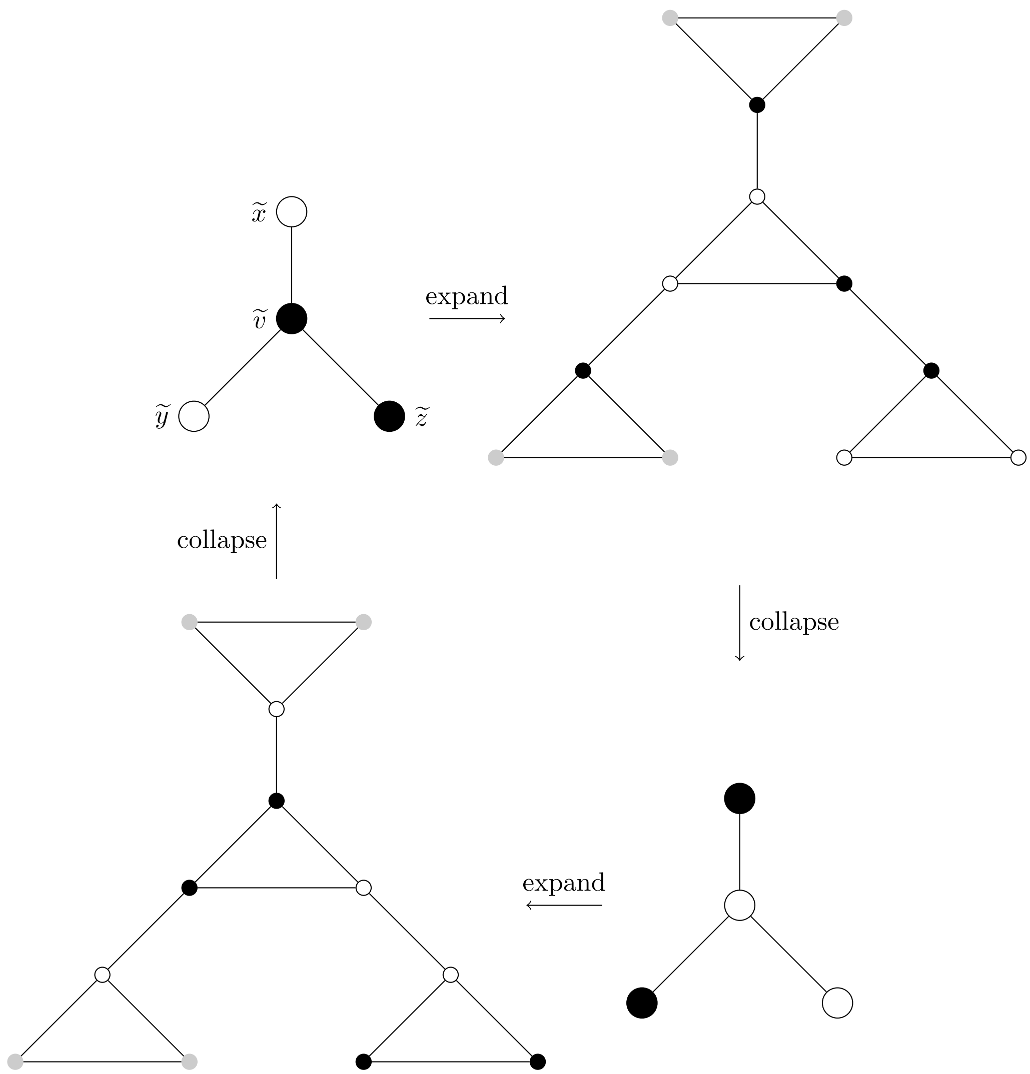


Figure 6: Expanding and collapsing, illustrating the labeling scheme for the new vertices. Label 1 is black, label 2 is white, gray nodes are unspecified, but in each such 3-cycle, one gray node is black (1) and one is white (2).

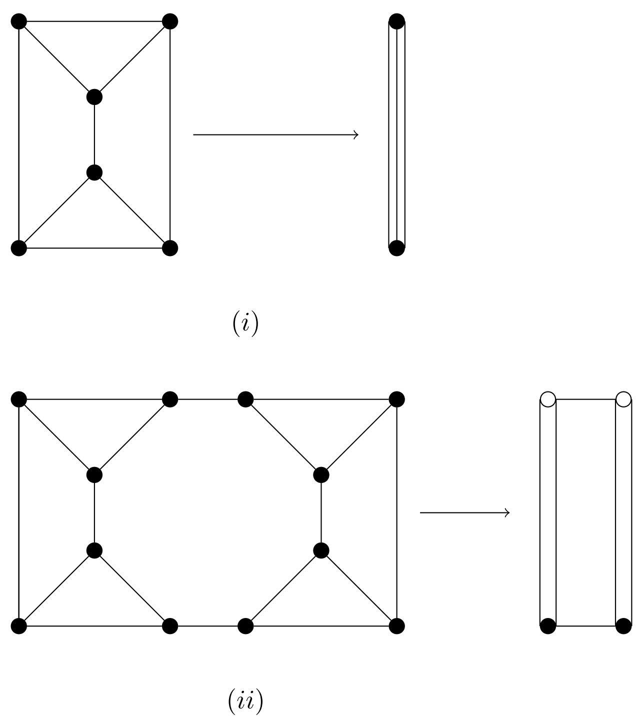


Figure 7: Illustration of the collapsing theorem. In (i) is depicted a graph whose collapse clearly cannot be $(2,2)$-NVL, and it's easy to see that the graph is not either. In (ii) is depicted a graph whose collapse (right) has been labeled (colored) to show that it is $(2,2)$-NVL. This is one of the two $(2,2;2,2)$-NL graphs on 12 vertices (see Figure 1 for the other, which can be obtained from K_4 in an analogous manner). The multigraph on the bottom right will be referred to as M_4 in the text.

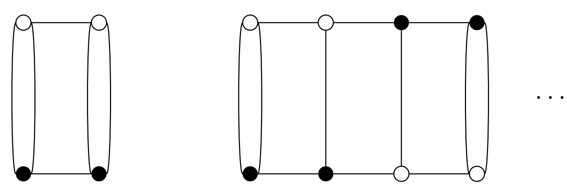
as in the reverse of Theorem 2.2. Consider the following two (not necessarily disjoint) sets:

$$\begin{aligned} \mathcal{S}_g &= \{(2,2) - \text{NVL multigraphs constructable from } K_4\} \\ \mathcal{S}_m &= \{(2,2) - \text{NVL multigraphs constructable from } M_4\}. \end{aligned}$$

The above discussion shows that the set of all $(2,2)$-NVL multigraphs is the union of $\mathcal{S}_g$ and $\mathcal{S}_m$. Is $\mathcal{S}_m \setminus \mathcal{S}_g = \{M_4\}$, or is there a $(2,2)$-NVL multigraph on more than 4 vertices that is not constructable from K_4?

Figure 8 shows all possible $(2,2)$-NVL multigraphs on 8 vertices. Note that clearly the one on the bottom right cannot be constructed from K_4 through only the addition of 4 vertices. Thus $\mathcal{S}_m \setminus \mathcal{S}_g \neq \{M_4\}$. We require M_4 as well as K_4 to construct all possible $(2,2)$-NVL multigraphs. Whether there are $(2,2)$-NVL graphs (with no multiple edges) that cannot be constructed from K_4 using only the addition of 4 vertices is at present unknown.

Note that Theorem 2.2 results in a much stronger characterization of $(2,2)$-NVL graphs than Theorem 2.1. We can obtain all $(2,2)$-NVL multigraphs from just two "atomic" multigraphs, K_4 and M_4. We do not have a characterization of the atomic multigraphs that generate the $(2,2;2,2)$-NL graphs. It is easy to see that the set of atoms is infinite, even if we restrict to connected graphs – just consider the following infinite set of atoms:



3. Conclusion

We have presented a new kind of labeled graph, where each neighborhood of a vertex "looks like" every other neighborhood. This definition of homogeneity puts severe constraints on the types of graphs that are possible, and we have illustrated this by considering the case of $(2,2;2,2)$ neighborhood homogeneous graphs.

There are many possible extensions of this work. A complete characterization of $(a,b;c,d)$-NL graphs is of interest. Certain constraints on the numbers are obvious, and discussed above, but it would be of interest to know whether all attainable choices of $(a,b;c,d)$ result in infinite families of graphs, and whether there are certain non-obvious choices that cannot be met in any graph. Also, how many non-isomorphic neighborhoods are there that make up the building blocks of an $(a,b;c,d)$-NL graph?

There are other ways one might consider to extend the concept of local homogeneity. For example, one might consider k-neighborhoods (all vertices of

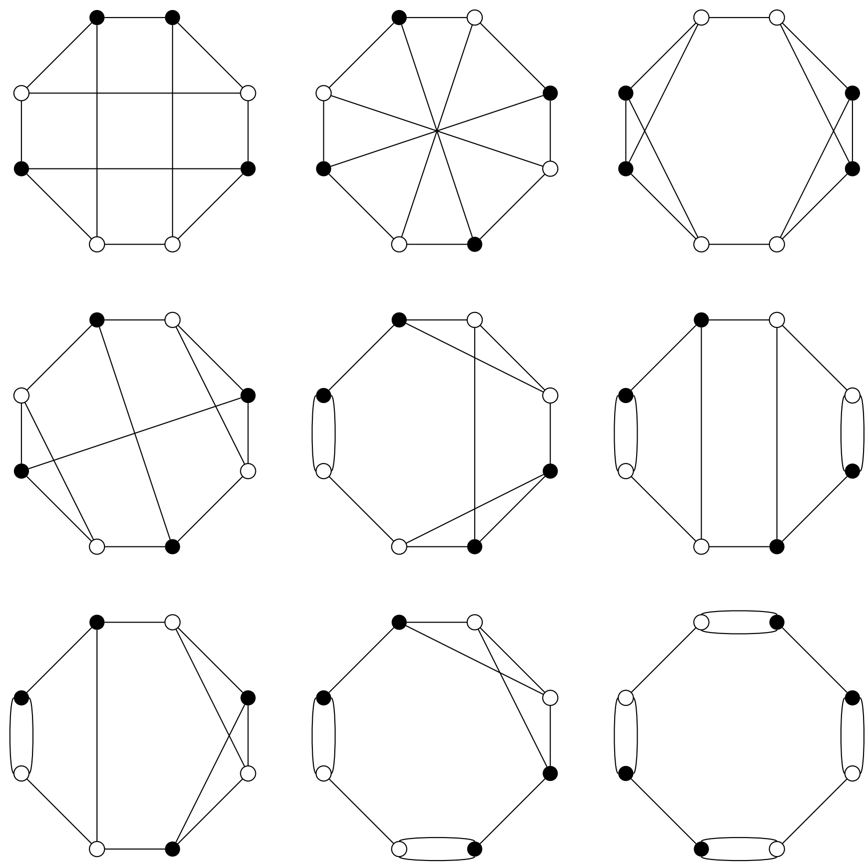


Figure 8: The 9 possible $(2,2)$-NVL multigraphs on 8 vertices. On each graph a vertex labeling is shown as solid and empty circles.

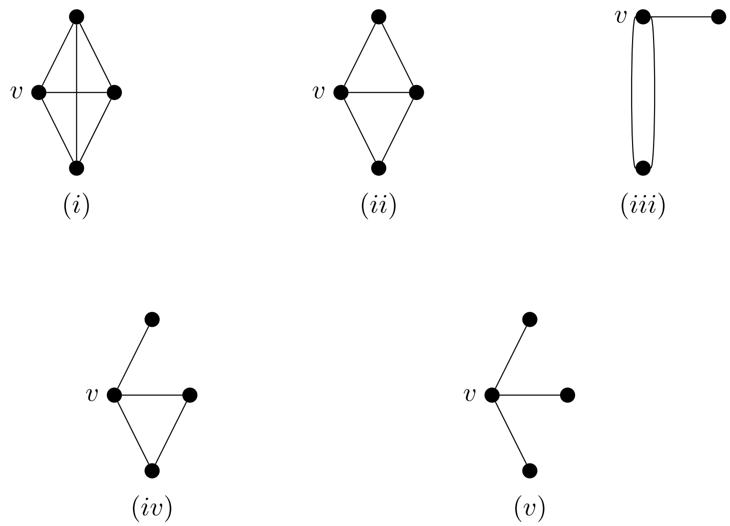


Figure 9: Five possible neighborhoods of a vertex v in the $(2,2)$-NVL multigraph considered in Theorem 2.2.

distance less than or equal to k) instead of 1-neighborhoods. Another condition to consider is to insist that the neighborhoods be isomorphic, as well as containing the same number of each label. For the $(2,2;2,2)$-NL graphs such as in Figure 1 this condition is naturally satisfied (although it is not if we allow multigraphs).

Another version of homogeneity that might be of interest in some applications is one in which the proportions of labels are constant across neighborhoods, rather than requiring numerical equality. For example we might only require that there are equal numbers of each label, without specifying the numbers. This would be closer to the statistical homogeneity considered in [5].

Of interest also is the extent to which inhomogeneity can be introduced locally. For example, is it possible for a graph to be $(a,b;c,d)$-NL everywhere except for the neighborhood of one vertex? If so, what are the constraints on how that neighborhood can deviate from homogeneity?

Appendix

Proof of Theorem 2.2

The possible neighborhoods of a vertex are shown in Figure 9. We will show that in each case, the 4 vertices in this neighborhood can be removed from G, edges reconnected or added as necessary, resulting in a new $(2,2)$-NVL multigraph. The labels will be referred to as "black" and "white" in keeping with the depictions (see, for example, Figure 10).

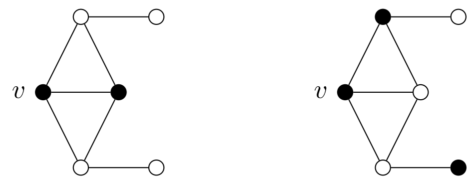


Figure 10: Possible labelings for the neighborhood of Figure 9 (ii), with two vertices added to indicate the coloring (labeling) of the vertices connecting the neighborhood to the rest of the graph.

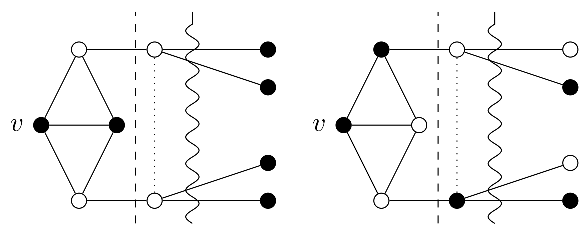


Figure 11: Extending the neighborhoods of Figure 10. The wiggly curve indicates the rest of the graph, and is meant to denote the fact that we don't really know the details of the structure to the right, beyond the color (label) of the vertices indicated. In particular, we don't know whether same-color vertices are distinct or what edges there are beyond the ones indicated. The dashed line indicates the edges to be cut to remove the 4 nodes, and the dotted line indicates an added edge to make the resulting graph $(2,2)$-NVL.

(i): If this is the neighborhood of v, then the neighborhood of v is a connected component of G that is a copy of K_4, and since $n > 4$ we can remove this connected component leaving a $(2,2)$-NVL graph.

(ii): There are two possible vertex labelings for this neighborhood (up to swapping the labels and a flip of the graph), as indicated in Figure 10. In this figure we have added the two vertices that connect this neighborhood to the rest of the graph. Note that in the case on the left, although the two added vertices have the same label, they must be distinct. Further, each of these are connected to two black vertices (these four vertices need not be distinct). Since none of these black vertices (not shown in the figure) connect to v and its four neighbors, we can remove this neighborhood and connect an edge between the two white neighbors without effecting their neighborhoods. The resulting graph is clearly $(2,2)$-NVL if the original one was. A similar analysis works for the other labeling in the Figure. See Figure 11, resulting in an edge between the white and black vertices on the right. Note that whether the four black vertices (in the left figure) or the pairs of black and white vertices (in the right figure) are distinct is irrelevant to the argument. In the rest of the cases where there is ambiguity of this type, it will always be the case that the different choices can be handled. We will show representative cases, or simply state that they have been checked, to reduce the number of pictures necessary in this paper.

This type of analysis is used throughout the proof. Note that the $(2,2)$-NVL

requirement puts very strong restrictions on what color (label) of vertex can be connected to the neighborhood under consideration, and where it can connect. This can result in a rather large number of possibilities, but as will be seen is always restricted enough for easy enumeration and checking.
(*iii*): Figures 12 and 13 depict the case for (*iii*). This is slightly more complicated than the case (*ii*); there are two sub-cases, each of which result in separate cases. As can be seen, however, for each possible labeling one or more edges (indicated by the dotted line or curves) can be added so that the resulting graph is still $(2,2)$-NVL.

Note that there is an equivalent condition as in the gray box, where the bottom white vertices are the same with a double edge, and the analysis of this case is essentially the same.
(*iv*): There are only a few possibilities to check for this case, some of which are illustrated in Figure 14. It is straightforward to check all possible such graphs. For example, in the top row, the final two black and two white nodes are not necessarily distinct, but it is easy to check that these cases all work.
(*v*): The final case is illustrated in Figure 15. We show only a few of the possible cases. The other cases can be easily checked by the reader.

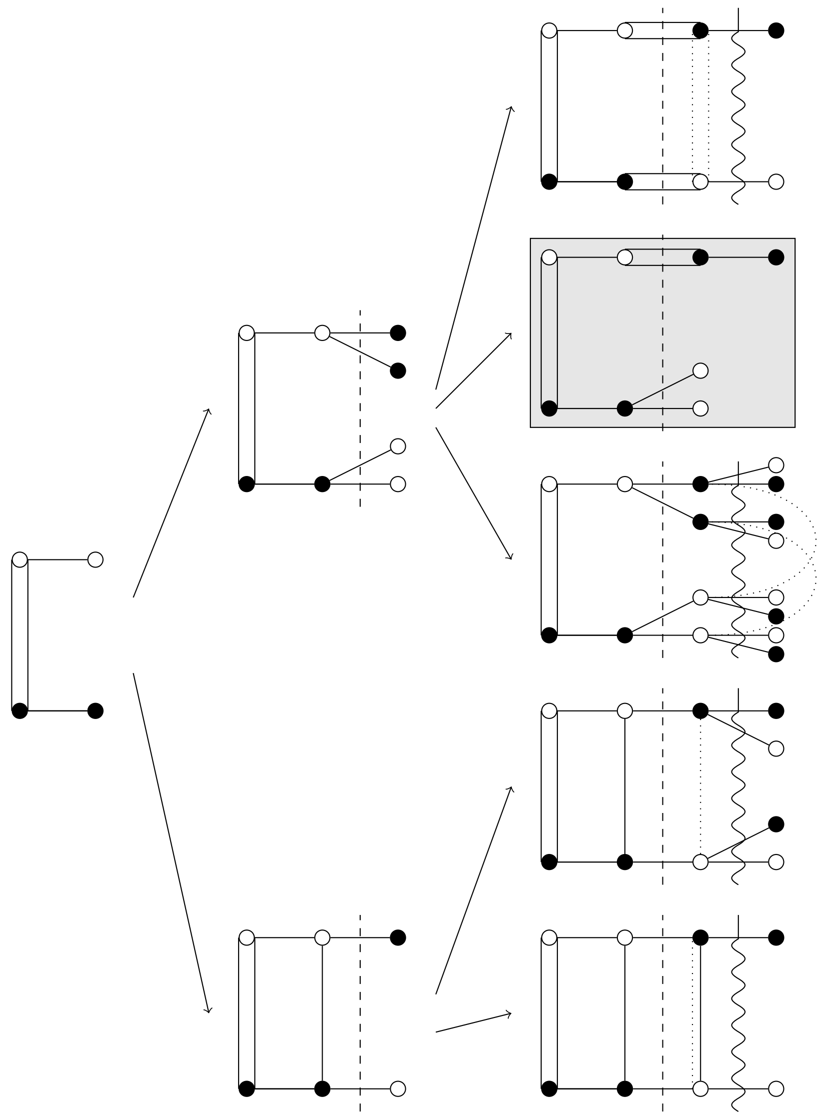


Figure 12: Extending the neighborhoods of Figure 9 (*iii*). The same convention is used for dashed, dotted and wiggly lines as in Figure 11. The condition indicated by the gray box is considered in Figure 13.

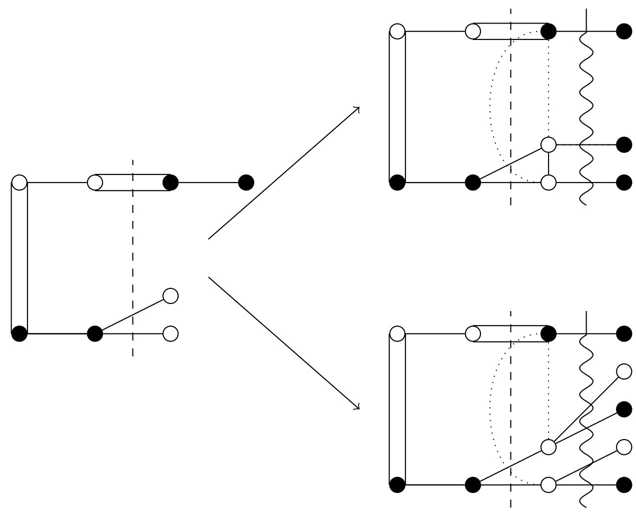


Figure 13: Continuation of the case (*iii*) considered in Figure 12 (gray box).

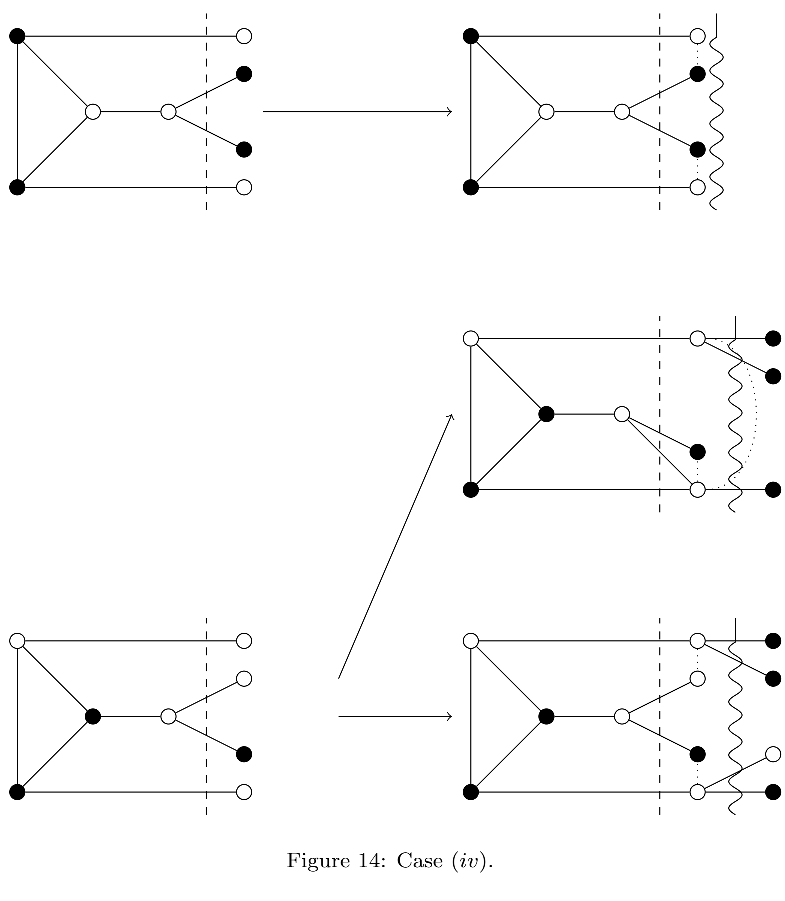


Figure 14: Case (*iv*).

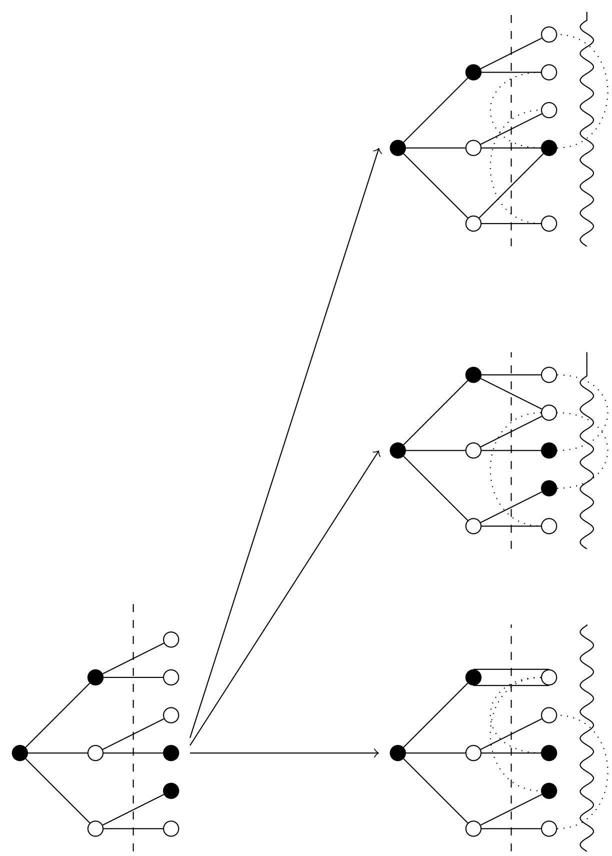


Figure 15: Case (v).

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