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The Generalized Spherical Homeomorphism Theorem for Digital Images

Lowell Abrams, Donniell E. Fishkind, and Carey E. Priebe*

***Abstract*—The spherical homeomorphism conjecture, proposed by Shattuck and Leahy in 2001, serves as the backbone of their algorithm to correct the topology of magnetic resonance images of the human cerebral cortex. Using a canonical image-thickening technique and the authors' previously proven "spherical homeomorphism theorem for surfaces," we formulate and prove a spherical homeomorphism theorem which is valid for all digital images when utilizing the (26,6)-connectivity rule.**

***Index Terms*—Digital topology, spherical homeomorphism.**

I. Introduction and Outline

The human cerebral cortex, viewed as closed at the brain stem, may be thought of as a surface topologically equivalent to a sphere. Due to noise and resolution issues, a magnetic resonance image approximating the cerebral cortex may fail to be spherical and, worse yet, may not even be a surface. Correcting the spherical topology is important for mapping the regions of the cerebral cortex by neurological and biological function, and much attention has been paid to this problem in the recent literature (see [2], for instance, and the references listed there). In [3], Shattuck and Leahy proposed a method to correct the spherical topology based on their Spherical Homeomorphism Conjecture, which conjectured that the boundary of a digital image is topologically spherical if and only if the "foreground" and "background" graphs associated with the image are trees. In [1], we showed that, subject to the condition that the boundary of the digital image is a surface, the conjecture is true. The idea of the proof was to consider the Euler characteristics for the boundaries of [the parts of the image represented by] the vertices and edges of the associated graphs and to then combine these into the global Euler characteristic for the boundary of the entire digital image.

In this paper, we show that if the boundary of a connected digital image is not a surface there is a canonical way to adjust it so as to yield a surface. Then we show that when the foreground and background graphs are constructed using the (26,6)-connectivity rule, the Spherical Homeomorphism Conjecture holds for the adjusted boundary.

In Section II, we show that the boundary of a connected digital image fails to be a surface precisely when the image contains one of three "forbidden" subimages, and that a simple thickening fixes these problems, rendering the boundary a surface. In Section III, we review the Spherical Homeomorphism Conjecture and the special case for which it was previously proved. We then formulate and prove a Spherical Homeomorphism Theorem which is true for all digital images when utilizing the (26,6)-connectivity rule.

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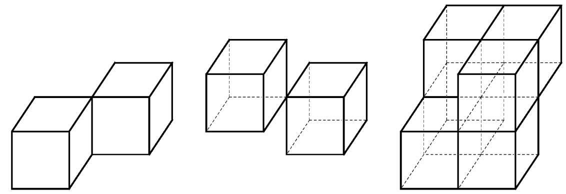

(a) (b) (c)

Fig. 1. The fundamental obstructions to being locally homeomorphic to a disk.

II. Thickening Nonsurfaces

We use the term *surface* to refer to a compact, connected subset of $\mathbb{R}^3$ that is locally homeomorphic to a disk. Since the boundaries of digital images, as we will define them, are always compact, and since connectivity in the context of the Spherical Homeomorphism Conjecture is inherent, the only obstruction to the boundary of a digital image being a surface is the requirement that it be locally homeomorphic to a disk. The main results of this section are Theorem 1, which characterizes such nonsurfaces, and Theorem 2, which provides a simple solution to the problem of this obstruction.

For any positive integers i, j, k, we define the i, j, kth voxel $v_{i,j,k}$ to be the unit cube $[i-1,i] \times [j-1,j] \times [k-1,k] \subseteq \mathbb{R}^3$. For any positive integers q, r, and s, we define

$$\Omega^{q,r,s} := \{v_{i,j,k} : 1 \leq i \leq q, 1 \leq j \leq r, 1 \leq k \leq s\}.$$

We refer to a subset $A \subseteq \Omega^{q,r,s}$ as a *digital image* and let $\widetilde{A}$ denote $\cup_{v \in A} v \subseteq \mathbb{R}^3$. We denote by $\partial\widetilde{A}$ the boundary of $\widetilde{A}$, and we write A^c for the complement of A in $\Omega^{q,r,s}$. If each $q, r, s \leq 2$, then we say A is an *elementary* digital image.

Suppose $A \subseteq \Omega^{q,r,s}$, $B \subseteq \Omega^{q',r',s'}$ such that each of q', r', and s' are, respectively, less than or equal to q, r, and s. We say B is a *subimage* of A if there exists nonnegative integers x_0, y_0, z_0 less than or equal to $q-q'$, $r-r'$, $s-s'$, respectively, such that for all positive integers i, j, k less than or equal to q', r', s', respectively, we have $v_{i,j,k} \in B$ if and only if $v_{x_0+i,y_0+j,z_0+k} \in A$.

We refer to the elementary digital images of Fig. 1(a)–(c) which are, respectively, in $\Omega^{2,2,1}$, $\Omega^{2,2,2}$, and $\Omega^{2,2,2}$, as *forbidden* digital images. Note that Fig. 1(b) and (c) are complements of each other. The boundaries of these three forbidden digital images are not locally homeomorphic to a disk due to zero-dimensional (0-D) or one-dimensional (1-D) identifications on their boundaries.

We say two digital images B, C are *equivalent* if there is an isometry (spatial rotation or reflection) from $\widetilde{B}$ onto $\widetilde{C}$. We extend the term "forbidden" to include any elementary digital image containing a subimage equivalent to a forbidden image. In particular, the elementary digital images in Fig. 2 which are marked with an $*$ are forbidden.

Theorem 1: For any digital image $A \subseteq \Omega^{q,r,s}$, $\partial\widetilde{A}$ is locally homeomorphic to a disk if and only if A contains no forbidden subimage.

Proof of Theorem 1: If A contains a forbidden subimage then $\partial\widetilde{A}$ is not locally homeomorphic to a disk at a 0-D or 1-D identification on the boundary of the forbidden subimage. Conversely, if there is a point on $\partial\widetilde{A}$ where $\partial\widetilde{A}$ is not locally homeomorphic to a disk then, for some elementary subimage E of image A, this point corresponds to a point on $\partial\widetilde{E}$ where $\partial\widetilde{E}$ is not locally homeomorphic to a disk. We only need to verify that this E is a forbidden digital image, and the result follows. Indeed, it is straightforward to enumerate all classes of equivalent elementary digital images; the numbers of such

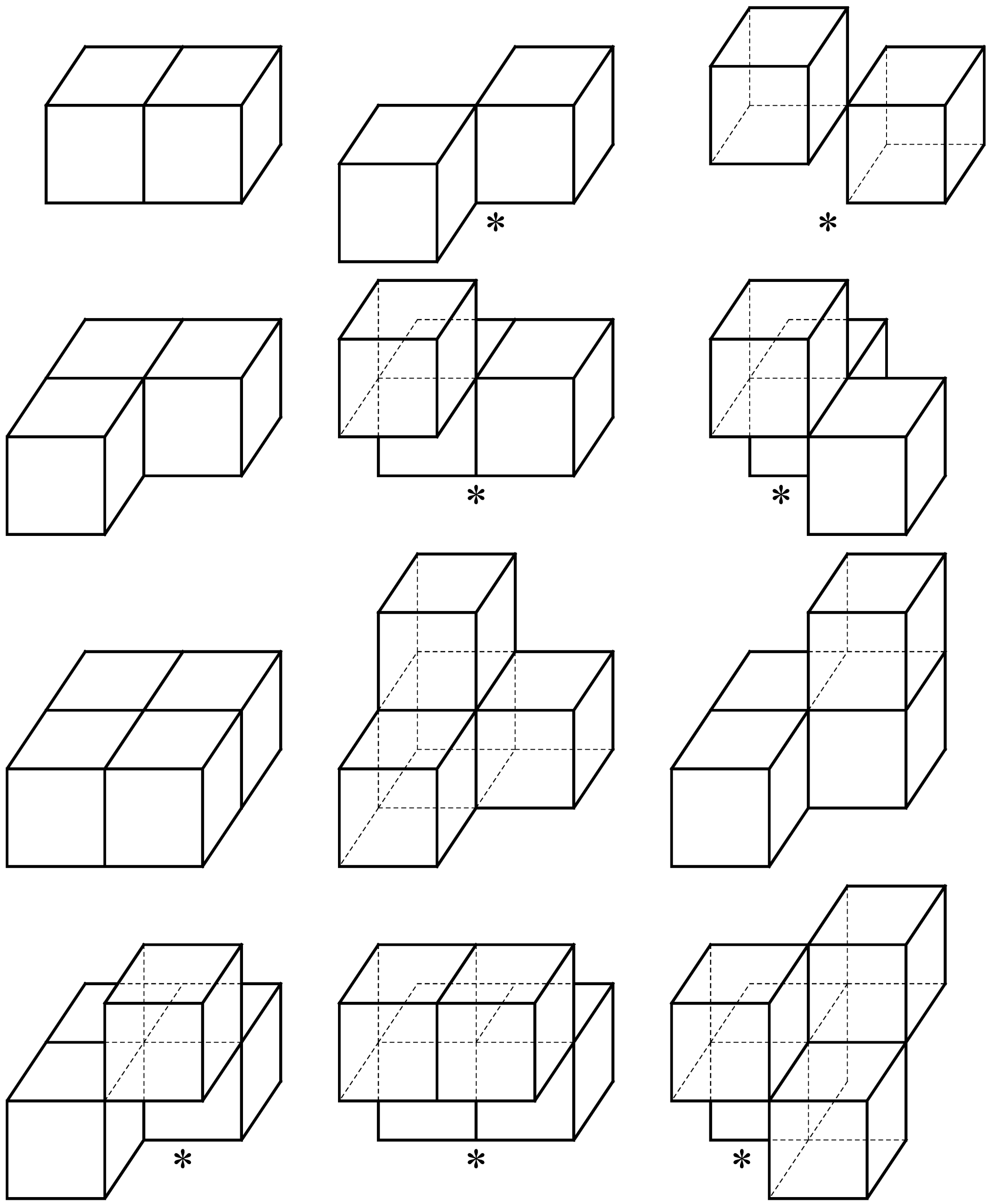

Fig. 2. The twelve elementary subimages with two, three, or four foreground voxels; those marked with an $*$ have a boundary which is not locally homeomorphic to a disk.

classes with 0, 1, 2, 3, 4, 5, 6, 7, 8 voxels, respectively, are 1, 1, 3, 3, 6, 3, 3, 1, 1. The 12 different classes with 2, 3, or 4 voxels are displayed in Fig. 2; all other classes are trivial or "dual" to these through complementation. By inspection, those classes E such that $\partial\widetilde{E}$ is not locally homeomorphic to a disk (which are marked with an $*$ in Fig. 2) are forbidden. $\square$

Let n be a positive integer greater than 2 and set $\epsilon = 1/n$. For any $\zeta := (x, y, z) \in \mathbb{R}^3$, we define $\zeta^\star$ to be the set $\{(x', y', z) \in \mathbb{R}^3 : |x' - x| \leq \epsilon, |y' - y| \leq \epsilon\}$; thus, $\zeta^\star$ is a two-dimensional (2-D)[1] $2\epsilon \times 2\epsilon$ square centered at ζ. Now, we define the *ϵ-thickening* of $\widetilde{A}$ to be $\widetilde{A}^\star := \cup_{\zeta \in \widetilde{A}} \zeta^\star$. The ϵ-thickenings of (the original three) forbidden images are displayed in Fig. 3; note that the boundary of each of these thickened images is locally homeomorphic to a disk.

[1]We could alternatively define $\zeta^\star$ by expanding about ζ in all three dimensions (without affecting any result in this manuscript), but the definition we use here greatly simplifies exposition later.

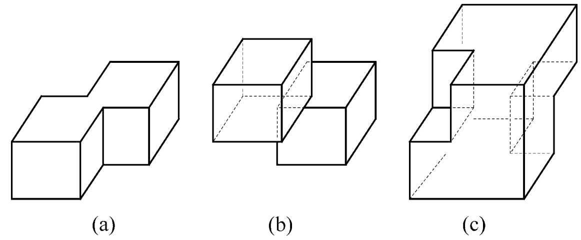

Fig. 3. Thickened forbidden images.

Theorem 2: For any digital image $A \subseteq \Omega^{q,r,s}$, $\partial\widetilde{A}^\star$ is locally homeomorphic to a disk. Thus, if $\partial\widetilde{A}^\star$ is connected then $\partial\widetilde{A}^\star$ is a surface.

Proof of Theorem 2: About each point in $\partial\widetilde{A}^{\star}$ there is an open neighborhood homeomorphic to an open neighborhood about an associated point in $\partial\widetilde{E}^{\star}$ for some elementary subimage E of A. An inspection of the respective ϵ-thickenings of all elementary images (whose classes are enumerated above) shows that each such ϵ-thickening has a boundary which is locally homeomorphic to a disk. Thus, $\partial\widetilde{E}^{\star}$ and $\partial\widetilde{A}^{\star}$ are locally homeomorphic to a disk. □

Informally, the miniscule expansion from $\widetilde{A}$ to $\widetilde{A}^{\star}$ swallows any 0-D or 1-D boundary identifications into the interior without creating any new 0-D or 1-D boundary identifications. In particular, this thickening naturally suits the (26,6)-connectivity rule discussed in Section III.

III. The Spherical Homeomorphism Theorem

Voxels $v, v' \in \Omega^{q,r,s}$, are said to be 26-*adjacent* if the intersection $v \cap v'$ (in $\mathbb{R}^3$) is nonempty, and are said to be 6-*adjacent* if $v \cap v'$ is 2-D. These notions of adjacency should be viewed as discrete relations on the finite set $\Omega^{q,r,s}$, relative to which a collection of voxels is either 6-connected or 6-disconnected and either 26-connected or 26-disconnected. At the same time, these notions reflect different strengths of connectivity of $v \cup v'$ as a subset of $\mathbb{R}^3$; 26-adjacency corresponds to connectivity of $v \cup v'$, while 6-adjacency corresponds to connectivity of the interior of $v \cup v'$.

For each $k = 1, 2, \ldots, s$, let L_k denote the kth *level* of $\Omega^{q,r,s}$, i.e., the set of voxels $v_{i,j,k}$ where i, j vary freely. Let a digital image $A \subseteq \Omega^{q,r,s}$ be given. We call each 26-connected component of voxels in $L_k \cap A$ a *foreground vertex*, and each 6-connected component of voxels in $L_k \cap A^c$ a *background vertex*. For each pair of foreground vertices W, W' satisfying $\widetilde{W} \cap \widetilde{W}' \neq \emptyset$, each connected component of $\widetilde{W} \cap \widetilde{W}'$ is called a *foreground edge with endpoints* W and W'. For each pair of background vertices W, W' satisfying $\widetilde{W} \cap \widetilde{W}' \neq \emptyset$, each connected component of the relative interior[2] of $\widetilde{W} \cap \widetilde{W}'$ is called a *background edge with endpoints* W and W'. The *foreground graph* of A (*background graph* of A, respectively) is the multi-graph $G_f(A)$ (respectively, $G_b(A)$) with vertex set consisting of all foreground vertices (resp., background vertices) and with edge set consisting of all foreground edges (resp., background edges).

We say the digital image $A \subseteq \Omega^{q,r,s}$ is *standard* if for every $v_{i,j,k} \in A$, we have $i \notin \{1, q\}$, $j \notin \{1, r\}$, $k \notin \{1, s\}$. The following is a slightly modified version of the Spherical Homeomorphism Conjecture of Shattuck and Leahy [3]:

Conjecture 3 (Spherical Homeomorphism Conjecture): If $A \subseteq \Omega^{q,r,s}$ is a standard digital image, then $\partial\widetilde{A}$ is topologically equivalent to a sphere if and only if both $G_f(A)$ and $G_b(A)$ are graph-theoretic trees.

For digital images A such that $\partial\widetilde{A}$ is not a surface, the conjecture needs to be modified; $\partial\widetilde{A}$ cannot be topologically equivalent to a sphere if $\partial\widetilde{A}$ is not a surface, whether or not $G_f(A)$ and $G_b(A)$ are trees. On the other hand, if $\partial\widetilde{A}$ is a surface then the conjecture is true; this is proved in [1].

Theorem 4 (Spherical Homeomorphism Theorem for Surfaces, A-F-P 2002): If $A \subseteq \Omega^{q,r,s}$ is a standard digital image such that $\partial\widetilde{A}$ is a surface, then $\partial\widetilde{A}$ is topologically equivalent to a sphere if and only if both $G_f(A)$ and $G_b(A)$ are graph-theoretic trees.

Note that if $\partial\widetilde{A}$ is a surface then A contains no forbidden subimages. From this it is easy to verify that, when $\partial\widetilde{A}$ is a surface, the type of connectivity (6 versus 26) used in defining the foreground and background graphs is immaterial.

We now present our main result.

Theorem 5: If $A \subseteq \Omega^{q,r,s}$ is a standard digital image then $\partial\widetilde{A}^{\star}$ is topologically equivalent to a sphere if and only if both $G_f(A)$ and $G_b(A)$ are graph-theoretic trees.

[2]This difference in definition between foreground and background edges reflects the difference in the respective notions of connectivity.

Proof of Theorem 5: Let $A \subseteq \Omega^{q,r,s}$ be a standard digital image and suppose first that $\partial\widetilde{A}^{\star}$ is connected; by Theorem 2, $\partial\widetilde{A}^{\star}$ is a surface. Consider the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ sending $(x, y, z) \rightarrow (nx, ny, z)$. Since $n = 1/\epsilon$, we have $T(\widetilde{A}^{\star}) = \widetilde{B}$ for some digital image $B \subseteq \Omega^{nq,nr,s}$. Moreover, $\partial\widetilde{B}$ is a surface of the same genus as $\partial\widetilde{A}^{\star}$. By the Spherical Homeomorphism Theorem for Surfaces, $\partial\widetilde{B}$, and hence $\partial\widetilde{A}^{\star}$ as well, are topologically equivalent to a sphere if and only if both $G_f(B)$ and $G_b(B)$ are trees. We next show that $G_f(A)$ and $G_b(A)$ are, respectively, isomorphic to $G_f(B)$ and $G_b(B)$; Theorem 5, in the case that $\partial\widetilde{A}^{\star}$ is connected, then follows.

Let $\tau: \Omega^{q,r,s} \rightarrow \mathbb{R}^3$ denote the map sending $C \mapsto \widetilde{C}$ where $C \subseteq \Omega^{q,r,s}$. The effect of the ϵ-thickening is that two voxels v, v' (in A, say) are 26-adjacent in $\Omega^{q,r,s}$ iff $T(\tau\{v, v'\}^{\star})$ is connected which, in turn, happens iff the interior of $T(\tau\{v, v'\}^{\star})$ is connected. Thus, the vertices and edges of $G_f(A)$, which are formed using 26-adjacency, correspond to the vertices and edges of $G_f(B)$. On the other hand, two voxels v, v' (in A^c, say) are 6-adjacent in $\Omega^{q,r,s}$ iff $\widetilde{\Omega}^{nq,nr,s} \setminus T\left(\tau\left(\{v, v'\}^c\right)^{\star}\right)$ is connected which, in turn, happens iff the interior of $\widetilde{\Omega}^{nq,nr,s} \setminus T\left(\tau\left(\{v, v'\}^c\right)\right)$ is connected. Thus, the vertices and edges of $G_b(A)$, which are formed using 6-adjacency, correspond to the vertices and edges of $G_b(B)$. These correspondences provide isomorphisms between $G_f(A)$ and $G_f(B)$ and between $G_b(A)$ and $G_b(B)$, respectively.

Finally, if $\partial\widetilde{A}^{\star}$ is not connected then it is not a surface, so $\partial\widetilde{B}$ is a disjoint union of surfaces. This implies that one of $G_f(B)$ and $G_b(B)$ is disconnected, hence one of them fails to be a tree, and thus $G_f(A)$ and $G_b(A)$ are not both trees. □

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