Problem 1.

(a) If we erode $F$ by a disk structuring element $B$ that is centered at the origin, we will obtain image $F \ominus B \subseteq F$. If we subtract this image from $F$, we will obtain the edges of $F$. Therefore, a useful morphological edge detector is

$$\Psi_{\text{edge}}(F) = F \setminus F \ominus B \subseteq F.$$ 

By taking the diameter of $B$ to be 1, we obtain the following result:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image.png}
\caption{Diagram illustrating the edge detection process.}
\end{figure}

(b) We can choose the diameter of $B$ to be equal to $2\epsilon$, where $\epsilon \ll 1$. Then, $\Psi_{\text{edge}}(F)$ will produce the same result as before but with thickness $\epsilon$. This will reduce the edge thickness to a thin line of width $\epsilon$.

Problem 2.

(a) Since $F$ is $B$-open, we have that $F = G \oplus B$ with $G = F \ominus B$. Then, $F \ominus B = (G \oplus B) \ominus B = G \bullet B$. Therefore, $F \ominus B$ is equivalent to the structural closing $G \bullet B$.

(b) Since $(\Psi_{\epsilon}, \Psi_{\delta})$ is an adjunction, we have that $\Psi_{\epsilon} \Psi_{\delta} \Psi_{\epsilon}(F) = \Psi_{\epsilon}(F)$ and $\Psi_{\delta} \Psi_{\epsilon} \Psi_{\delta}(F) = \Psi_{\delta}(F)$. In this case, $\Psi_{\epsilon} \Psi_{\delta} \Psi_{\epsilon} \Psi_{\delta} \Psi_{\epsilon}(F) = \Psi_{\epsilon} \Psi_{\delta}(F) \supseteq F$. 


**Problem 3.** We have the following results:

Clearly, $F_1 \subseteq F_2$, $F_3$, $F_4$, $F_3 \subseteq F_2$, and $F_4 \subseteq F_2$, but we cannot order $F_3$ and $F_4$.

**Problem 4.** Let $S$ be a $2 \times 2$ square structuring element that contains the origin. Then, $F \ominus S$ contains the centers of all $2 \times 2$ squares in $F$. In this case, 

\[ F_1 = \Psi_1(F) = F \setminus R_B(F \ominus S | F) \quad \text{and} \quad F_2 = \Psi_2(F) = R_B(F \ominus S | F), \]

where $R_B$ is the binary reconstruction operator.

**Problem 5.** The best morphological operator for this case is given by

\[ \Psi(G) = \bigcup_{n \in Q^+} [G \ominus nB \setminus G \ominus (n + 1)B], \]

where $Q^+ = \{ n \geq 0 \mid P_N(n) < P_F(n) \}$.

From the given pattern spectra, it is clear that $Q^+ = \{2, 5, 6\}$. Therefore, 

\[ \Psi(G) = (G \ominus 2B \setminus G \ominus 3B) \cup (G \ominus 5B \setminus G \ominus 6B) \cup (G \ominus 6B \setminus G \ominus 7B) \]

\[ = (G \ominus 2B \setminus G \ominus 3B) \cup (G \ominus 5B \setminus G \ominus 7B). \]
Problem 6.

(a) For $0 \leq t \leq 0.5$, we have the situation depicted below.

Note that $|F \circ tB| = |F_1| + |F_2|$, where

$|F_1| = 2 - 3 \left( t^2 - \frac{\pi t^2}{4} \right) = 2 - 3t^2 + \frac{3}{4}\pi t^2$

$|F_2| = 1 - 2 \left( t^2 - \frac{\pi t^2}{4} \right) = 1 - 2t^2 + \frac{1}{2}\pi t^2.$

Therefore,

$S_F(t) = 2 - 3t^2 + \frac{3}{4}\pi t^2 + 1 - 2t^2 + \frac{1}{2}\pi t^2 = 3 - 5t^2 + \frac{5}{4}\pi t^2 = 3 - 1.075t^2.$

Hence

$S_F(t) = 3 - 1.075t^2,$  for  $0 \leq t \leq 0.5,$

which leads to the following plot:
(b) At \( t = 0 \), \( S_F(t) \) gives the area of the object, which equals to 3 in this case. As \( t \) increases, \( S_F(t) \) is a continuous function of \( t \) and decreases monotonically. This indicates that \( F \) contains no objects smaller than a disk of radius \( 1/2 \) (otherwise a discontinuity will occur).

**Problem 7.**

(a) An operator that distributes over intersections.

(b) An operator that is increasing, extensive and idempotent.

(c) An operator that is increasing and idempotent.

(d) The operator composed of the translation-invariant erosion followed by the adjunct translation invariant dilation.

(e) The reconstruction operator when the marker shape is the structural opening of image \( F \) with a structuring element \( A \).