MORPHOLOGICAL IMAGE ANALYSIS VII

Binary Morphological Filtering

- A set operator $\Psi(\bullet)$ is said to be a **morphological filter**, if:

  \[
  \begin{align*}
  \Psi(\bullet) & \text{ is increasing; i.e., } F_1 \subseteq F_2 \Rightarrow \Psi(F_1) \subseteq \Psi(F_2) \\
  \text{and} \\
  \Psi(\bullet) & \text{ is idempotent; i.e., } \Psi(\Psi(F)) = \Psi(F)
  \end{align*}
  \]

- The first property (increasing) is also satisfied by a linear filter with *positive* impulse response!

- The second property (idempotent) is clearly satisfied by all *ideal linear filters* (e.g., take an ideal lowpass filter with frequency response $H(u,v) = 1$, for $-u_c \leq u \leq u_c$ and $-v_c \leq v \leq v_c$, and $H(u,v) = 0$, otherwise; then, $H(u,v)H(u,v)F(u,v) = H(u,v)F(u,v)$, for all frequencies $(u,v)$, which leads to: $h \ast (h \ast f) = h \ast f$).

**Two Basic Morphological Filters**

- The **opening** $F \circ B$ is a morphological filter.

- The **closing** $F \bullet B$ is a morphological filter.

- However, the erosion and dilation are not morphological filters (they are not idempotent).
Composition of Morphological Filters

We can build morphological filters from other morphological filters by composition.

If $\Psi_1$ and $\Psi_2$ are two morphological filters such that $\Psi_1 \subseteq \Psi_2$ or $\Psi_2 \subseteq \Psi_1$, then:

$$\Psi_2 \Psi_1, \Psi_1 \Psi_2, \Psi_1 \Psi_2 \Psi_1,$$  and  $$\Psi_2 \Psi_1 \Psi_2$$

are morphological filters as well.

Based on these compositions, the operators (recall that $F \bigcirc kB \subseteq F \bullet kB$):

$$\Pi_k(F) = (F \bigcirc kB) \bullet kB$$
$$P_k(F) = (F \bullet kB) \bigcirc kB$$

are also morphological filters; they are known as alternating filters (AF) (they alternate between opening and closing).

The operators:

$$T_k(F) = ((F \bullet kB) \bigcirc kB) \bullet kB$$
$$\Sigma_k(F) = ((F \bigcirc kB) \bullet kB) \bigcirc kB$$

are morphological filters as well.

Finally, we can compose alternating filters to form another class of morphological filters known as alternating sequential filters (ASF), given by:

$$M_k(F) = \Pi_k \Pi_{k-1} \cdots \Pi_1(F)$$
$$N_k(F) = P_k P_{k-1} \cdots P_1(F)$$

The ASFs are more preferable in practice, as compared to the AFs, since, for a given value of $k$, they gradually filter-out shape components by means of structuring elements $B, 2B, 3B, \ldots, kB$, as opposed to the case of AFs where shape components are filtered-out at once by means of structuring element $kB$. 
**Morphological Pattern Restoration**

- In many image processing and analysis applications, image data $F$ is corrupt by noise and clutter.

- In this case, we are interested in designing an operator $\Psi(\bullet)$ which, when applied on the corrupted data $G$, optimally recovers $F$.

- To solve this problem by means of mathematical morphology, we usually assume that $F$ is corrupted by the so-called **union-intersection noise**, in which case:

\[
G = (F \cap N_1^c) \cup N_2
\]

where $N_1$ is the **intersection noise** and $N_2$ is the **union noise**.

**Example**

![Example Image](image-url)
The Germ-Grain Noise Model

- In the previous slide we have used a specific model for noise that is known as the germ-grain noise model.

- Points $p_i, i = 1, 2, ..., \text{(the germs)}$ are randomly distributed on the image plane.

- At each point $p_i$ a shape (set) $G_i$ (the grain) is centered.

- Then, the germ-grain noise model is given by

$$N = \bigcup_{i \geq 1} G_i + p_i$$

**Example**

- When the germs are distributed by means of a 2-D Poisson model, and the grains are random shapes independently and identically distributed (i.i.d.) to a random shape $G$, then the resulting model is known as the Boolean model.

- In the above example, the germs are distributed as a 2-D Poisson process and the grains are disks with random radii; therefore, the image depicts a realization of a Boolean model of random disks.

- In the discrete case, the germs are usually distributed according to a 2-D Bernoulli process.
Union-Intersection Noise

- In the union-intersection noise:
  \[ G = (F \cap N_1^c) \cup N_2 \]
  we frequently take \( N_1 \) and \( N_2 \) to be (usually i.i.d.) germ-grain processes.

- Notice that the union-intersection noise becomes identical to salt-and-pepper noise when the grains in the underlying germ-grain processes are single points.

- Therefore, the union-intersection-noise is a generalization of the salt-and-pepper noise to allow modeling structure in the underlying grains.

Example

Example images showing grains of different radii:  
- Grains: disks of random radii \( \leq 5 \)  
- Grains: disks of radii \( = 0 \)
Morphological Pattern Restoration

- The problem of morphological pattern restoration amounts to designing an image operator $\Psi(\bullet)$ such that:

$$\hat{F} = \Psi(G) = \Psi((F \cap N_1^c) \cup N_2) = F$$

That is, $\hat{F}$ is a good approximation to $F$.

- The effectiveness of obtaining $F$ from $G$ by means of $\Psi$ strongly depends on certain “non-overlapping” characteristics of the noise free image $F$ and the noise components $N_1$ and $N_2$ typical to the particular filtering problem at hand.

- It also depends on the effectiveness of $\Psi$ in discriminating between these characteristics.

- For example, if we assume that the noise-free image $F$ is “lowpass” whereas the noise components $N_1$, $N_2$ are sufficiently “highpass” (in the sense that the “frequency bands” of $F$ and $N_1$, $N_2$ do not overlap), then an “ideal lowpass” filter with appropriate “cutoff frequency” will perfectly reconstruct $F$ from $G$.

- To design such a filter, we certainly need to define what we mean by “lowpass,” “highpass,” “frequency bands,” etc. and make the regularizing assumption that the degradation equation

$$G = (F \cap N_1^c) \cup N_2$$

is limited to “lowpass” images $F$.

- Notice however that $\Psi$ should be such that $\Psi(F) = F$, for all “lowpass” images, so that $\Psi$ does not affect “lowpass” images when $N_1$ and $N_2$ are both empty.
"Bandlimited" Noise

- To be more specific, notice here that the noise components $N_1$ and $N_2$ are usually bandlimited, in the sense that:

$$P_{N_1:B}(n) = P_{N_2:B}(n) = 0, \text{ for } n \geq n_0$$

where $P_{N:B}(n)$ is the pattern spectrum.

- This simply says that the "size" of the noise components is small, limited between 0 and $n_0 - 1$.

Examples

![Diagram showing examples with $n_0 = 2$ and $n_0 = 4$.]
Smooth Shapes

If we consider “size” as being the discriminating factor between \( F \) and \( N_1, N_2 \), then the noise-free image \( F \) should be “bandlimited” as well, in the sense that:

\[
P_{F,B}(n) = 0, \text{ for } -n_0 \leq n \leq n_0 - 1
\]

As we said before,

\[
P_{F,B}(n) = 0, \text{ for } 0 \leq n \leq n_0 - 1 \iff F = F \cap n_0 B
\]

which shows that the pattern spectrum at sizes \( 0 \leq n \leq n_0 - 1 \) is zero if and only if the shape \( F \) is \( n_0 B \)–open.

It can also be shown that

\[
P_{F,B}(n) = 0, \text{ for } -n_0 \leq n < 0 \iff F = F \bullet n_0 B
\]

which shows that the pattern spectrum at sizes \( -n_0 \leq n < 0 \) is zero if and only if the shape \( F \) is \( n_0 B \)–closed.

A shape \( F \) for which \( P_{F,B}(n) = 0, \text{ for } -n_0 \leq n \leq n_0 - 1 \), is called smooth to a degree \( n_0 \) (and relative to the structuring element \( B \)).

Notice that a shape \( F \) is smooth to a degree \( n_0 \) if and only if it is both \( n_0 B \)–open and \( n_0 B \)–closed.