MORPHOLOGICAL IMAGE ANALYSIS III

Secondary Binary Morphological Operators

A set operator \( \Psi(\bullet) \) that is:

- **Increasing**: i.e., \( F_1 \subseteq F_2 \Rightarrow \Psi(F_1) \subseteq \Psi(F_2) \)
- **Anti-extensive**: i.e., \( \Psi(F) \subseteq F \)
- **Idempotent**: i.e., \( \Psi(\Psi(F)) = \Psi(F) \)

is called an **opening**.

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- **Idempotent**: i.e., \( \Psi(\Psi(F)) = \Psi(F) \)

is called a **closing**.

If \( (\Psi, \delta) \) is an adjunction, then \( \Psi \circ \delta(\bullet) \) is an opening and \( \Psi \circ \delta(\bullet) \) is a closing.

If \( \{ \Psi_i \} \) is a collection of openings, then \( \Psi(\bullet) = \bigcup_i \Psi_i(\bullet) \) is an opening as well; i.e., *unions of openings is also an opening*!

If \( \{ \Psi_i \} \) is a collection of closings, then \( \Psi(\bullet) = \bigcap_i \Psi_i(\bullet) \) is a closing as well; i.e., *intersection of closings is also a closing*!
Binary Structural Opening

- The (binary) *structural opening* of an image $F$ with a structuring element $B$ is given by:

$$F \circ B = (F \ominus B) \oplus B$$

- It can be shown that:

$$F \circ B = \bigcup \{B + h | (B + h) \subseteq F\}$$

- This formula suggests that the structural opening of a set (shape) $F$ by a structuring element $B$ is the union of all translated structuring elements $B + h$ that fit inside $F$ (geometric interpretation of the structural opening).

- Opening a shape $F$ by a structuring element $B$ eliminates all components of $F$ that are “smaller” than $B$, in the sense that they cannot contain any translated replica of $B$! Therefore, the structural opening acts as a **shape filter**.
Binary Structural Opening

Example

\[ B: \bullet \]

\[ rB = \{rb \mid b \in B \} \]
Binary Structural Opening

- The structural opening is a **smoothing filter**!

- The amount and type of smoothing is determined by the shape and size of the structuring element used.

- It attempts to undo the effect of erosion $F \ominus B$ by applying the adjunct dilation $(F \ominus B) \oplus B$.

- The structural opening approximates a shape from inside, since

\[
F \circ B \subseteq F
\]
Binary Structural Closing

- The (binary) structural closing of an image $F$ with a structuring element $B$ is given by:

$$F \circ B = (F \oplus B) \ominus B$$

- It can be shown that:

$$F \circ B = \{u \mid u \in \bar{B} + h \Rightarrow (\bar{B} + h) \cap F \neq \emptyset\}$$

- This formula suggests that a point $u$ belongs to the structural closing of a set (shape) $F$ by a structuring element $B$ if all translated structuring elements $\bar{B} + h$ that contain $u$ intersect $F$ (geometric interpretation of the closing).
Binary Structural Closing

Example

\[ B: \bullet \]
\[ rB = \{rb \mid b \in B\} \]
Binary Structural Closing

- The structural closing is a smoothing filter!

- The amount and type of smoothing is determined by the shape and size of the structuring element used.

- It attempts to undo the effect of dilation $F \oplus B$ by applying the adjunct erosion $(F \oplus B) \ominus B$.

- The structural closing approximates a shape from outside, since $F \bullet B \supseteq F$. 
### Properties of Binary Structural Opening

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F \circ {h} = F )</td>
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<tr>
<td>Translation invariance</td>
<td>((F + h) \circ B = (F \circ B) + h)</td>
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<tr>
<td>( F \circ (B + h) = F \circ B )</td>
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<td>Increasingness with respect to shape</td>
<td>( F_1 \subseteq F_2 \Rightarrow F_1 \circ B \subseteq F_2 \circ B )</td>
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<tr>
<td>Homogeneity</td>
<td>( rF \circ rB = r(F \circ B) )</td>
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<tr>
<td>Anti-extensivity</td>
<td>( F \circ B \subseteq F )</td>
</tr>
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<td>Idempotence</td>
<td>((F \circ B) \circ B = F \circ B )</td>
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<td>Duality</td>
<td>( F \circ B = (F^c \bullet \overline{B})^c )</td>
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### Properties of Binary Structural Closing

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Some Advanced Properties of Structural Opening and Closing

- A set $F$ is called $B$–**open** if $F \bigcirc B = F$.

- A set $F$ is called $B$–**closed** if $F \bullet B = F$.

**Property:** A set $F$ is $B$–open if and only if $F = G \oplus B$ for some set $G$.

**Proof:** Assume first that $F$ is $B$–open. Then

$$F = F \bigcirc B = (F \bigcirc B) \oplus B = G \oplus B$$

Where $G = F \bigtriangledown B$. Assume now that $F = G \oplus B$, for some set $G$. Then

$$F \bigcirc B = (G \bigoplus B) \bigcirc B = ((G \oplus B) \bigoplus B) \oplus B = (G \bullet B) \oplus B \supseteq G \bigcirc B = F.$$ 

Also, $F \bigcirc B \subseteq F$, due to the anti-extensivity property of the opening; therefore, since $F \bigcirc B \supseteq F$ and $F \bigcirc B \subseteq F$, we get $F \bigcirc B = F$, which shows that $F$ is $B$–open.

**Property:** A set $F$ is $B$–closed if and only if $F = G \bigtriangledown B$ for some set $G$.

**Property:** Let $A$ and $B$ be two structuring elements such that $A$ is $B$–open. Then,

$$F \bigcirc A \subseteq F \bigcirc B$$

$$(F \bigcirc A) \bigcirc B = (F \bigcirc B) \bigcirc A = F \bigcirc A$$

$$F \bullet A \supseteq F \bullet B$$

$$(F \bullet A) \bullet B = (F \bullet B) \bullet A = F \bullet A$$
Morphological Shape Smoothing

Example 1

Example 2

Example 3
Binary Erosion-Dilation-Opening-Closing Example 1

Henri Matisse, *Nu Bleu III*, 1952
Binary Erosion-Dilation-Opening-Closing Example 2

\[ B: \begin{array}{c}
\end{array} \quad \begin{array}{c}
F
\end{array} \]

\[ F \odot B \quad F \oplus B \]

\[ F \odot B \quad F \oplus B \]

\[ F \odot B \quad F \oplus B \]
Binary Erosion-Dilation-Opening-Closing Example 3

\[ B: - \quad \text{and} \quad F \]

\[ F \ominus B \quad \text{and} \quad F \oplus B \]

\[ F \odot B \quad \text{and} \quad F \bullet B \]
Remark

- From the previous examples it is clear that the choice of the structuring element is very important here.

Here is another example

- Most of the time, the structuring element is chosen by the algorithm designer in a way that makes most sense in practical applications.

- However, many researchers have identified the need to build-up an approach that allows the automatic specification of the structuring element that “best fits” a particular application.