Linear and Non-Linear Scale-Spaces I

- The main idea behind *scale-spaces* is to summarize an image at different scales, where all image details are still present at the finest scale, while they start disappearing at increasingly coarser scales.

- The rationale behind this is twofold:
  - *Creations of a natural visual hierarchy.* The importance of structures is reflected in their survival time under these “summarization” operations. The motivation at the heart of this approach is that progressively increased processing will cause unimportant details to blend in with the background, thus forcing attention on structures and their interrelations with more prominent features.
  
  - *Combining descriptions across scales yields additional information.* The information at coarse scales can be used to re-interpret information that might be ambivalent at the myopic pixel-level. Coarse scale may be used to *identify* structures and fine scales may be used to *localize* them.

- Given an image $f(x,y)$, we are interested in a family of scale-space operators $\{K_\sigma; \sigma \geq 0\}$ that will map the image $f$ to a unique function $f(x,y,\sigma)$, such that

$$f(x,y,\sigma) = K_\sigma[f(x,y)] \quad \text{and} \quad f(x,y,0) = f(x,y)$$

- Each of the derived images $f(x,y,\sigma)$ can be thought of as a “blurred” version of the original, with $\sigma$ measuring the amount of blurring.

- The function $f(x,y,\sigma)$ is called the *scale-space* associated with image $f$, generated by the family $\{K_\sigma; \sigma \geq 0\}$ of scale-space operators.
Linear Scale-Spaces

- In this case, the scale-space operator $K_\sigma$ is given by

$$K_\sigma[f(x)] = k_\sigma(x) * f(x),$$

where $*$ denotes convolution and $k_\sigma(x)$ is the point spread function of a linear smoothing filter, parameterized by $\sigma$.

- For simplicity, we limit our exposition here to the 1-D case, but extensions to higher dimensions run along similar lines.

- In scale-space theory, we are most often interested in the evolution of the zero-crossings of the second order derivative

$$E(x, \sigma) = \frac{\partial^2 f(x, \sigma)}{\partial x^2} = \frac{\partial^2 k_\sigma(x)}{\partial x^2} * f(x)$$

- We are interested in finding kernels $k_\sigma(x)$ for which zero-crossings are never created as scale increases.

- In terms of the underlying signal, we are interested in kernels that never create extrema or edges that are “spurious” in the sense that they are artifacts of the convolution process rather than features of the underlying signal.

- Furthermore, as the scale parameter increases these extrema tend to collapse and annihilate each other, creating a tree in scale-space that captures the structure of the image in terms of edges and bright or dark blobs.
Linear Scale-Spaces

- Yuille and Poggio have imposed a number of straightforward conditions:
  
  1. The filter at different scales is a simple rescaling of a fixed profile:

\[
k_\sigma(x) = \frac{1}{\sigma} \kappa(x / \sigma)
\]

  2. The kernel is symmetric about its center and this is independent of \( \sigma \). Otherwise, zero-crossings of a step edge would change their position with changes in scale.

  3. The filter vanishes at infinity (\( |x| \to \infty \)) and recovers the whole image at sufficient small scales:

\[
\lim_{\sigma \to 0^+} k_\sigma(x) = \delta(x)
\]

  4. Contours of the zero crossings of

\[
E(x, \sigma) = \frac{\partial^2 f(x, \sigma)}{\partial x^2} = \frac{\partial^2 k_\sigma(x)}{\partial x^2} * f(x)
\]

should never be “created” as \( \sigma \) increases. More precisely, at points \( (x_0, \sigma_0) \) on such contours for which

\[
E(x_0, \sigma_0) = 0 \quad \text{and} \quad \frac{\partial E}{\partial x}(x_0, \sigma_0) = 0
\]

holds, the following monotonicity condition should be satisfied:

\[
\frac{\partial E}{\partial \sigma} \frac{\partial^2 E}{\partial x^2} > 0
\]
Linear Gaussian Scale-Spaces

- It can be shown that conditions 1, 2, and 4 imply that
  \[
  \frac{1}{\sigma} \frac{\partial k_\sigma(x)}{\partial \sigma} = \frac{\gamma}{\theta} \frac{\partial^2 k_\sigma(x)}{\partial x^2}
  \]
  where \( \gamma / \theta > 0 \) is a constant.

- By a change of variables \( \tau = \sigma^2 / 2 \) and \( \rho^2 = 2 \gamma / \theta \), we obtain
  \[
  \frac{\partial k_\tau(x)}{\partial \tau} = \frac{\rho^2}{2} \frac{\partial^2 k_\tau(x)}{\partial x^2}
  \]
  which is the standard diffusion (or heat) equation!!

- The only solution to this equation that satisfies condition 3 is the Gaussian kernel
  \[
  k_\sigma(x) = G_\sigma(x) = \frac{1}{\sqrt{2\pi\rho^2\tau}} e^{-x^2/2\rho^2\tau} = \frac{1}{\sqrt{\pi \rho \sigma}} e^{-(x/\rho\sigma)^2}
  \]

- In one dimension, the Gaussian is the only filter that never creates zero-crossings of the second derivative as the scale increases.
- In two dimensions, the Gaussian is the only filter that never creates zero-crossings of the Laplacian as the scale increases.
Linear Gaussian Scale-Spaces

- An obvious disadvantage of Gaussian smoothing is the fact that it does not only smooth noise, but also blurs important features such as edges and, thus, makes them harder to identify. Since Gaussian smoothing is designed to be completely uncommitted, it cannot take into account any a-priori information on structures that are worth being preserved (or even enhanced).

- Linear diffusion filtering dislocates edges when moving from finer to coarser scales. So, structures that are identified at a coarse scale do not give the right location and have to be traced back to the original image.

- In 2-D, although new zero-crossings cannot be created as $\sigma$ increases, they can both split and merge so that the number of zero crossing contours can both increase and decrease.

- Diffusion in 2-D can create new extrema.

- From an image processing point of view this last remark is particularly disappointing.
Perona-Malik Anisotropic Diffusion

- One of the most important aspects of the axiomatic approach to scale-spaces is the prominent role played by the diffusion equation, an equation that keeps cropping up in all sorts of guise.

- However, the limitations of this evolution equation are well known.

- The most important limitation, in terms of practical usefulness, is that although noise is gradually removed and the overall structure is simplified, edges are irrevocably blurred and displaced.

- In an attempt to remedy some of the shortcomings of scale-spaces based on the diffusion equation, Perona and Malik extended the scope of their investigation to include non-linear evolution equations.

- They enumerated three principles that, in their view, were crucial prerequisites:

  1. **Causality:** Preventing the generation of “spurious detail.”

  2. **Immediate localization:** At each resolution, region boundaries should coincide with semantically meaningful edges (i.e., edges should not shift away from the region boundaries to which they semantically belong).

  3. **Localized smoothing:** Smoothing should preferentially occur inside semantically meaningful regions, and not across their boundaries.
Perona-Malik Anisotropic Diffusion

- Perona and Malik proposed to use a diffusion of the form

\[
\frac{\partial f(x,y,\sigma)}{\partial \sigma} = \text{div} \left( c(\| \nabla f(x,y,\sigma) \|) \nabla f(x,y,\sigma) \right)
\]

where

\[
\nabla f(x,y,\sigma) = \frac{\partial f(x,y,\sigma)}{\partial x} \hat{x} + \frac{\partial f(x,y,\sigma)}{\partial y} \hat{y},
\]

\(c(\bullet)\) is a symmetric, bell-shaped function, such as

\[
c(v) = \frac{1}{1 + (v/K)^2} \quad \text{or} \quad c(v) = e^{-(v/K)^2/2},
\]

and “div” denotes divergence, given by

\[
\text{div} \left[ f_1(x,y)\hat{x} + f_2(x,y)\hat{y} \right] = \frac{\partial f_1(x,y)}{\partial x} + \frac{\partial f_2(x,y)}{\partial y}
\]

- This equation applies anisotropic diffusion on the original image \(f(x,y,0) = f(x,y)\), guided by the gradient magnitude \(\| \nabla f(x,y,\sigma) \|\).

- Large values of \(\| \nabla f(x,y,\sigma) \|\) are indicative of region boundaries, and one wants the diffusion at those locations to slow down \((c \rightarrow 0)\).

- Small values of \(\| \nabla f(x,y,\sigma) \|\) are indicative of the interior of regions, and one wants the diffusion at those locations to increase \((c \rightarrow 1)\).

- It can be shown, however, that this nonlinear anisotropic diffusion equation is characterized by a rather unstable behavior.

- Nevertheless, researchers in image analysis have paid little attention to this problem and they have frequently obtained surprisingly good and stable results.
Morel-Lions Anisotropic Diffusion

- Considerable progress was made when work by a French group, headed by Morel and Lions, showed how these equations could be modified so that they allow the so-called viscosity solutions.

- Viscosity solutions represent a generalized form of PDE solutions for which differentiability conditions are relaxed so that merely continuous functions qualify as candidate solutions.

- In essence, the French group concentrated their efforts on the following basic model:

\[
\frac{\partial f(x,y,\sigma)}{\partial \sigma} = g(\| \sigma * f(x,y,\sigma) \|) \| \nabla f(x,y,\sigma) \| \text{div} \left( \frac{\nabla f(x,y,\sigma)}{\| \nabla f(x,y,\sigma) \|} \right)
\]

where

\[
G_\sigma(x,y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\left(x^2+y^2\right)/2\sigma}
\]

- Although, this formulation looks slightly daunting, a closer look reveals a simple geometric intuition behind it:

1. First of all, it is easy to check that introducing a new orthonormal (local) coordinate system \((\xi, \eta)\), where \(\eta\) is oriented along the gradient \(\nabla f\) while \(\xi\) is orthogonal to it, allows us to re-express the differential operator in a simpler form that represents diffusion where smoothing occurs along lines of equal gray value and not across them. As a consequence, edges delineating salient regions will be spared.

2. Function \(g(\bullet)\) controls the speed of the diffusion and is chosen to be a smoothly decreasing positive function such that \(g(0)=1\) and \(g(u) \to 0\), as \(u \to \infty\). As a consequence, diffusion near edges is slowed down, providing extra protection.
**Example: Restoration properties of diffusion filters.**

(a) Original noisy image.  
(b) Linear Gaussian diffusion.  
(c) Perona-Malik diffusion.  
(d) Morel-Lions diffusion.

**Remarks:**

1. Linear diffusion filtering is capable of removing noise, but we have to pay a price: the image becomes completely blurred. Edges get smoothed and become dislocated (*correspondence problem*) so that they are harder to identify.

2. In the case of Perona-Malik diffusion, edges are hardly affected, so that we have no correspondence problems, which are characteristic of linear diffusion. However, the drastically reduced diffusivity at edges is also responsible for the drawback that noise at edges is preserved.

3. The Morel-Lions diffusion shares the advantages of both methods: it combines good noise eliminating properties of linear diffusion with the stable edge structure of non-linear diffusion.
**Example:** Image segmentation using diffusion.

(a) Original MRI image.

(b) Result of nonlinear diffusion.

(c) Segmented image in (a).

(d) Segmented image in (b).