MORPHOLOGICAL IMAGE ANALYSIS II

Binary Morphological Operators

- We choose to deal with shape (image) operators that are **distribute** over set unions and set intersections.

- This is a type of “linearity.” In the case of the linear image processors that we studied in ECE 520.414/614, the image operators were chosen to distribute over addition.

- Any shape operator $\Psi_\varepsilon(\bullet)$ such that

  $$\Psi_\varepsilon(F_1 \cap F_2) = \Psi_\varepsilon(F_1) \cap \Psi_\varepsilon(F_2)$$

  for every pair of images $F_1, F_2$ is called a (binary) **erosion**.

- Any shape operator $\Psi_\delta(\bullet)$ such that

  $$\Psi_\delta(F_1 \cup F_2) = \Psi_\delta(F_1) \cup \Psi_\delta(F_2)$$

  for every pair of images $F_1, F_2$ is called a (binary) **dilation**.

- Erosions and dilations are the most elementary operators of mathematical morphology.

- We can design more complicated **morphological operators** by combining erosions and dilations.
Increasing and Translation Invariant Operators

- A set operator \( \Psi(\bullet) \) is **increasing** if \( F_1 \subseteq F_2 \) implies that \( \Psi(F_1) \subseteq \Psi(F_2) \).

- An increasing operator \( \Psi \) forbids an object \( F_1 \) that is occluded by an object \( F_2 \) to become visible after processing; i.e., \( \Psi(F_1) \) will still be occluded by \( \Psi(F_2) \).

- Both erosions and dilations are increasing operators!

- If \( F + h \) denotes the translated set \( \{ u + h \mid u \in F \} \),

then a set operator \( \Psi(\bullet) \) is **translation invariant** if

\[
\Psi(F + h) = \Psi(F) + h
\]

for every translation \( h \).
Translation Invariant Erosion

DISTRIBUTIVITY
\[ \Psi_\varepsilon(F_1 \cap F_2) = \Psi_\varepsilon(F_1) \cap \Psi_\varepsilon(F_2) \]

TRANSLATION INVARIANCE
\[ \Psi_\varepsilon(F + h) = \Psi_\varepsilon(F) + h \]

\[ \Psi_\varepsilon(F) = \bigcap_{b \in B} F - b = F \ominus B \]

- \( B \) is a subset of the two-dimensional space and is called the **structuring element**.

- In set theory, the translation invariant erosion is also known as the **Minkowski subtraction**.
Translation Invariant Erosion

- It can be shown that:

\[ F \ominus B = \{ h \mid (B + h) \subseteq F \} \]

- This formula suggests that the erosion of a set (shape) \( F \) by a structuring element \( B \) comprises all points \( h \) such that the structuring element \( B \) located at \( h \) fits entirely inside \( F \) (geometric interpretation of erosion).

- Eroding a shape \( F \) by a structuring element \( B \) (that contains the origin) has the effect of “shrinking” the shape in a manner determined by the shape and size of \( B \).
Translation Invariant Erosion

Example

\[ B : \bullet \]

\[ rB = \{ rb \mid b \in B \} \]
Translation Invariant Dilation

- $B$ is a subset of the two-dimensional space and is called the **structuring element**.

- In set theory, the translation invariant dilation is also known as the *Minkowski addition*. 

\[
\Psi_\delta(F \cup F_2) = \Psi_\delta(F_1) \cup \Psi_\delta(F_2) \\
\Psi_\delta(F + h) = \Psi_\delta(F) + h
\]

\[
\Psi_\delta(F) = \bigcup_{b \in B} F + b = F \oplus B
\]
Translation Invariant Dilation

- It can be shown that:

\[
F \oplus B = \{ h \mid (\tilde{B} + h) \cap F \neq \emptyset \}
\]

where \( \tilde{B} = \{-b \mid b \in B\} \) is the \textbf{reflection} of \( B \) with respect to the origin.

- This formula suggests that the dilation of a set (shape) \( F \) by a structuring element \( B \) comprises all points \( h \) such that the structuring element \( \tilde{B} \) located at \( h \) hits (intersects) \( F \) (geometric interpretation of dilation).

- Dilating a shape \( F \) by a structuring element \( B \) (that contains the origin) has the effect of “expanding” the shape in a manner determined by the shape and size of \( B \).
Translation Invariant Dilation

Example

\[ rB = \{ rb \mid b \in B \} \]
## Properties of Erosion

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where \( rF = \{ ru | u \in F \} \) is the scaled version of \( F \).

- A set operator \( \Psi(\bullet) \) is called an **anti-extensive operator** if

\[
\Psi(F) \subseteq F, \text{ for every } F
\]
# Properties of Dilation

\[
F \oplus \{h\} = F + h
\]

\[
F \oplus B = B \oplus F
\]

Commutativity

\[
(F + h) \oplus B = F \oplus (B + h) = (F \oplus B) + h
\]

Translation invariance

\[
F \ominus B_1 \subseteq F \ominus B_2 \text{ if } B_1 \subseteq B_2
\]

Increasingness with respect to structuring element

\[
F \ominus (B_1 \cup B_2) = (F \ominus B_1) \cup (F \ominus B_2)
\]

Parallel composition

\[
(F_1 \cup F_2) \ominus B = (F_1 \ominus B) \cup (F_2 \ominus B)
\]

Distributivity of union

\[
F \ominus (B_1 \cap B_2) \subseteq (F \ominus B_1) \cap (F \ominus B_2)
\]

Parallel composition inequality

\[
(F \ominus B_1) \ominus B_2 = F \ominus (B_1 \ominus B_2)
\]

Serial composition

\[
F_1 \subseteq F_2 \Rightarrow F_1 \ominus B \subseteq F_2 \ominus B
\]

Increasingness with respect to shape

\[
rF \ominus rB = r(F \ominus B)
\]

Homogeneity

\[
F \ominus B \supseteq F \text{ if } B \text{ contains the origin}
\]

Extensivity

\[
F \ominus B = (F^c \ominus \bar{B})^c
\]

Duality

where \( rF = \{ ru | u \in F \} \).

- A set operator \( \Psi(\bullet) \) is called an **extensive operator** if

\[
\Psi(F) \supseteq F, \text{ for every } F
\]
Adjunctions

- For the translation invariant erosion and dilation we have the following property:

\[
F \oplus B \subseteq G \iff F \subseteq G \ominus B
\]

- Notice that this is another form of duality between erosions and dilations.

- In fact, this property can be viewed as an “inverse” property: We can say that the erosion is the “inverse” of dilation in the sense that the above equation is satisfied.

- Notice however that the erosion is not the exact inverse of dilation: i.e.,

\[
G = F \oplus B \text{ does not imply that } F = G \ominus B
\]

- In mathematical morphology, every pair of set operators \((\Psi_\epsilon, \Psi_\delta)\) that satisfy the property

\[
\Psi_\delta(F) \subseteq G \iff F \subseteq \Psi_\epsilon(G)
\]

is called an adjunction. Therefore \((F \ominus B, F \oplus B)\) is an adjunction.

- It can be shown that if \((\Psi_\epsilon, \Psi_\delta)\) is an adjunction, then \(\Psi_\epsilon\) is an erosion and \(\Psi_\delta\) is a dilation.

- It can also be shown that if \((\Psi_\epsilon, \Psi_\delta)\) is an adjunction then:

\[
\begin{array}{c|c}
\Psi_\epsilon \Psi_\delta(F) \supseteq F & \Psi_\delta \Psi_\epsilon(F) \subseteq F \\
\Psi_\epsilon \Psi_\delta \Psi_\epsilon(F) = \Psi_\epsilon(F) & \Psi_\delta \Psi_\epsilon \Psi_\delta(F) = \Psi_\delta(F) \\
\Psi_\epsilon(F) = \bigcup \{ G \mid \Psi_\delta(G) \subseteq F \} & \Psi_\delta(F) = \bigcap \{ G \mid F \subseteq \Psi_\epsilon(G) \}
\end{array}
\]
Discrete Examples of Binary Erosions and Dilations

**Example 1:** This example illustrates an instance of the dilation operation. The coordinate system we use for all the examples is (row, column):

\[ B = \{(0,0), (0,1)\} \]

\[ F = \{(0,1), (1,1), (2,1), (2,2), (3,0)\} \]

\[ F \oplus B = \{(0,1), (1,1), (2,1), (2,2), (3,0), (0,2), (1,2), (2,2), (2,3), (3,1)\} \]
Example 2: This example shows that when the origin is not in the structuring element $B$, it may happen that the dilation of $F$ by $B$ has nothing in common with $F$:

![Diagram of Example 2](image-url)
Example 3: This example illustrates an instance of erosion:

$$B$$

$$F$$

$$F \ominus B$$
Example 4: This example illustrates how eroding with a structuring element which does not contain the origin can lead to a result, which has nothing in common with the set being eroded:

![Diagram of Example 4](image-url)