Hybrid Formal Power Series and Their Application to Realization Theory of Hybrid Systems

Mihály Petreczky
Centrum voor Wiskunde en Informatica (CWI)
P.O.Box 94079, 1090GB Amsterdam, The Netherlands
M.Petreczky@cwi.nl

Abstract

The paper presents the abstract framework of hybrid formal power series. Hybrid formal power series are analogous to non-commutative formal power series. Formal power series are widely used in control systems theory. In particular, theory of formal power series is the main tool for solving the realization problem for linear and bilinear control systems. The theory of hybrid formal power series developed in this paper plays a similar role in realization theory of hybrid systems. The paper develops theory of rational hybrid formal power series and their representations. The relevance of the abstract theory is demonstrated by presenting an application of the theory to solving the realization problem for linear and bilinear hybrid systems.

1 Introduction

Realization theory is one of central topics of systems theory. It studies the relationship between classes of control systems and classes of input/output behaviors. It also provides procedures for constructing a (possibly minimal) system of a certain class generating the specified input/output behavior. Realization theory helps to understand such important system theoretic properties as observability and controllability. Apart from its theoretical relevance, realization theory has the potential of being applied for developing control and identification methods, as development of linear systems theory has demonstrated.

The current paper develops an abstract framework which can be applied to realization theory of certain classes of hybrid systems. The abstract framework presented in the paper is analogous to the well-known framework of rational formal power series representations. Rational formal power series representations are widely used in systems theory, especially in realization theory of various classes of systems. There are many works on application of formal power series in realization theory, see for examples [8, 16, 15, 7]. Rational formal power series were also used for realization theory of switched systems, see [10, 13, 11].

In this paper we shall develop the theory of \textit{rational hybrid formal power series}. A \textit{hybrid power series} is essentially a pair consisting of a formal power series and a discrete input-output map. We will study families of such hybrid...
power series. A family of hybrid formal power series is *rational* if it admits a *hybrid representation*. The notion of hybrid representation is analogous to the notion of rational formal power series representation. Roughly speaking, a hybrid representation is a composition of several rational formal power series representations with a finite Moore-automaton. Since hybrid representations contain both discrete and continuous components, they seem to be a potentially useful tool for studying hybrid systems. The theory of hybrid formal power series presented in this paper relies very much on the classical theory of rational formal power series [15, 16, 2] and automata theory [5, 6]. In fact, it combines the two theories. The main questions will be the following.

**Existence of a hybrid representation** When does such a collection of hybrid power series admit a *hybrid representation*?

**Minimality of hybrid representation** What is the smallest possible hybrid representation of a family of hybrid formal power series? How can such hybrid representations be characterized? Is there always a smallest possible hybrid representation of a family of hybrid formal power series? Is such a minimal hybrid representation unique?

**Partial realization theory** How to construct a hybrid representation for a family of hybrid formal power series using only finite number of data?

The results obtained for rational hybrid representations are very similar to those of rational formal power series and finite automata. Let us formulate the main results on hybrid formal power series in an informal way.

**Existence of a hybrid representation** A family of hybrid formal power series has a hybrid representation, i.e., it is rational if and only if the corresponding family of classical formal power series has a rational representation, i.e., it is rational and the corresponding family of discrete input-output maps has a realization by a finite Moore-automaton.

**Minimality of hybrid representations** If a family of hybrid formal power series has a hybrid representation, then it has a minimal hybrid representation. A hybrid representation is minimal if and only if it is reachable and observable. Any two minimal hybrid representations of the same family of hybrid formal power series are isomorphic. Minimality, observability and reachability can be checked algorithmically. Any hybrid representation can be transformed to a minimal one and the transformation can be done by an algorithm.

**Partial realization theory** If the number of available data points is big enough and the family of hybrid formal power series is finite, then it is possible to construct a minimal hybrid representation of the family of hybrid formal power series from finitely many data points. The precise conditions for the number of data points are similar to the conditions in partial realization theory of linear and bilinear systems.

We will motivate the study of hybrid formal power series by applying the theory to the following two classes of hybrid systems: bilateral hybrid systems and linear hybrid systems. A linear (bilinear) hybrid system is a hybrid system such that the continuous dynamics at each location is determined by a
continuous time linear (bilinear) control system and the system switches from one discrete location to another whenever an external discrete input event takes place. The automaton specifying the discrete-state transition is assumed to be deterministic. Discrete events act as discrete inputs, one can specify arbitrary sequence of them arriving at any time instant. There are no guards and the reset maps are assumed to be linear. The inputs of a linear (bilinear) hybrid system are of two types. Piecewise-continuous inputs are fed to the linear (bilinear) system belonging to the current discrete location. Timed sequences of discrete events determine the relative arrival times and relative order of external events which trigger transition of discrete states. The outputs of the linear (bilinear) hybrid system consist of the continuous outputs of the underlying linear (bilinear) systems and the discrete outputs of the discrete states.

Realization theory for both linear and bilinear systems was investigated in the recent papers [9, 12]. Let us recall the realization problem for linear and bilinear hybrid systems.

1. \textit{Reduction to a minimal realization} Consider a linear (bilinear) hybrid system $H$, and a subset of its input-output maps $\Phi$. Find a minimal linear (bilinear) hybrid system which realizes $\Phi$.

2. \textit{Existence of a realization} Find necessary and sufficient condition for existence of a linear (bilinear) hybrid system realizing a specified set of input-output maps.

3. \textit{Partial realization} Find a procedure for constructing a linear (bilinear) hybrid system realization of a set of input-output maps from finite data.

The following results were presented in [12, 9].

- A linear (bilinear) hybrid system is a minimal realization of a set of input-output maps if and only if it is observable and semi-reachable. Minimal linear (bilinear) hybrid systems which realize a given set of input-output maps are unique up to isomorphism. Each linear (bilinear) hybrid system $H$ realizing a set of input-output maps $\Phi$ can be transformed to a minimal realization of $\Phi$.

- A set of input/output maps is realizable by a linear hybrid system if and only if it has a \textit{hybrid kernel representation}, the rank of its Hankel-matrix is finite, the discrete parts of the input/output maps are realizable by a finite Moore-automaton and certain other finiteness conditions hold. A set of input/output maps is realizable by a bilinear hybrid system if and only if it has a \textit{hybrid Fliess-series expansion}, the rank of its Hankel-matrix is finite and the discrete parts of the input/output maps are realizable by a finite Moore-automaton. There is a procedure to construct the linear (bilinear) hybrid system realization from the columns of the Hankel-matrix, and this procedure yields a minimal realization.

- There exists a procedure which constructs a linear (bilinear) hybrid system realization from finite data. Under certain conditions, similar to those for linear and bilinear systems, this realization is a minimal realization of the specified input-output maps.
The results described above are indeed very similar to those for hybrid formal power series. This is not a coincidence, in fact, the results announced above will be proven again in this paper by using theory of hybrid formal power series. It turns out that there is one-to-one correspondence between linear and bilinear hybrid systems and hybrid representations of certain families of hybrid formal power series. This correspondence will enable us to reduce the realization problem for linear and bilinear hybrid systems to the problem of existence and minimality of hybrid representations for a certain family of hybrid formal power series. Moreover, such system theoretic properties of hybrid systems as observability, semi-reachability and minimality have their counterparts in hybrid representations. That is, there is one-to-one correspondence between reachable, observable, minimal hybrid representations and semi-reachable, observable, minimal linear and bilinear hybrid systems. Thus, theory of hybrid formal power series can be used to characterise minimality of linear and bilinear hybrid systems. It can be also used to derive partial realization theory of linear and bilinear hybrid systems.

Compared to the direct approach the use of hybrid formal power series helps to avoid unnecessary repetition of proofs and concepts. It also results in a much more elegant and concise treatment of realization theory for linear and bilinear hybrid systems.

The author hopes that the proposed framework will turn out to be useful not only for hybrid systems but for other classes of control systems such as multidimensional control systems studied in [1].

The outline of the paper is the following. The first section, Section 2, sets up some notation which will be used throughout the paper. Section 4 contains the necessary results on formal power series. The material of this section was already presented in more detail in [12, 9, 13]. Section 3 presents a summary on realization theory of finite Moore-automata. The material of this section was already presented in [12, 9]. Section 5 is the main section of the paper. It presents theory of hybrid formal power series. Section 6 contains the definition and some basic properties of hybrid systems. There are a number of slightly different definitions of hybrid systems. In Section 6 we presented a version which is the most suitable for the purpose of the current paper. Section 7 describes realization theory of linear hybrid systems. A more direct approach to realization theory of linear hybrid systems was already presented in [12]. Section 8 presents realization theory of bilinear hybrid systems. In [9] presents a more direct approach to realization theory of bilinear hybrid systems.

A more detailed presentation of the material of this paper can be found in [14]

2 Preliminaries

For an interval $A \subseteq \mathbb{R}$ and for a suitable set $X$ denote by $PC(A, X)$ the set of piecewise-continuous maps from $A$ to $X$, i.e., maps which have at most finitely many points of discontinuity on any bounded interval and at any point of discontinuity the left-hand and the right-hand side limits exist and are finite. For a set $\Sigma$ denote by $\Sigma^*$ the set of finite strings of elements of $\Sigma$. For $w = a_1a_2\ldots a_k \in \Sigma^*$, $a_1, a_2, \ldots, a_k \in \Sigma$ the length of $w$ is denoted by $|w|$, i.e. $|w| = k$. The empty sequence is denoted by $\epsilon$. The length of $\epsilon$
is zero: $|\epsilon| = 0$. Let $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. The concatenation of two strings $v = v_1 \cdots v_k, w = w_1 \cdots w_m \in \Sigma^+$ is the string $vw = v_1 \cdots v_kw_1 \cdots w_m$. We denote by $w^k$ the string $w \cdots w$. The word $w^0$ is just the empty word $\epsilon$. Denote by $k$-times by $T$ the set $[0, +\infty) \subseteq \mathbb{R}$. Denote by $\mathbb{N}$ the set of natural numbers including 0. Denote by $F(A,B)$ the set of all functions from the set $A$ to the set $B$. For any two sets $A,B$, define the functions $\Pi_A : A \times B \rightarrow A$ and $\Pi_B : A \times B \rightarrow B$ by $\Pi_A(a,b) = a$ and $\Pi_B(a,b) = b$. By abuse of notation we will denote any constant function $f : T \rightarrow A$ by its value. That is, if $f(t) = a \in A$ for all $t \in T$, then $f$ will be denoted by $a$. For any function $f$ the range of $f$ will be denoted by $\text{Im}f$. If $A,B$ are two sets, then the set $(A \times B)^*$ will be identified with the set $\{(u,w) \in A^* \times B^* \mid |u| = |w|\}$. For any set $A$ we will denote by $\text{card}(A)$ the cardinality of $A$. For any two sets $J, X$ an indexed subset of $X$ with the index set $J$ is simply a map $Z : J \rightarrow X$, denoted by $Z = \{a_j \in X \mid j \in J\}$, where $a_j = Z(j), j \in J$. Let $f : A \times (B \times C)^+ \rightarrow D$. Then for each $a \in A$, $w \in B^+$ we define the function $f(a,w,:) \in C^{[w]} \rightarrow D$ by $f(a,w,:) \in C^{[w]}$. By abuse of notation we denote $f(a,w,:) \in C^{[w]}$ by $f(a,w,v)$. Denote by $\mathbb{N}^k$ the set of $k$ tuples of non-negative integers. If $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$, then $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \ldots, \beta_m) \in \mathbb{N}^{k+m}$. Let $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^p$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. We define $D^\alpha \phi$ by

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{dt_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{dt_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_k}}{dt_k^{\alpha_k}} \phi(t_1, t_2, \ldots, t_k)|_{t_1 = t_2 = \cdots = t_k = 0}.$$

For each $f : T \rightarrow A$, $A$ an arbitrary set, and for each $\tau \in T$ denote by $\text{Shift}_\tau(f)$ the map

$$\text{Shift}_\tau(f) : T \ni t \mapsto f(\tau + t)$$

### 3 Finite Moore-automaton

A finite Moore-automaton is a tuple $A = (Q, \Gamma, O, \delta, \lambda)$ where $Q, \Gamma$ are finite sets, $\delta : Q \times \Gamma \rightarrow Q$, $\lambda : Q \rightarrow O$. The set $Q$ is called the state-space, $O$ is called the output space and $\Gamma$ is called the input space. The function $\delta$ is called the state-transition map and the function $\lambda$ is called the readout map. Denote by $\text{card}(A)$ the cardinality of the state-space $Q$ of $A$, i.e. $\text{card}(A) = \text{card}(Q)$. Define the functions $\tilde{\delta} : Q \times \Gamma^* \rightarrow Q$ and $\tilde{\lambda} : Q \times \Gamma^* \rightarrow O$ as follows. Let $\tilde{\delta}(q,\epsilon) = q$ and

$$\tilde{\delta}(q,w\gamma) = \delta(\tilde{\delta}(q,w),\gamma), w \in \Gamma^*, \gamma \in \Gamma$$

Let $\tilde{\lambda}(q,w) = \lambda(\tilde{\delta}(q,w)), w \in \Gamma^*$. By abuse of notation we will denote $\tilde{\delta}$ and $\tilde{\lambda}$ simply by $\delta$ and $\lambda$ respectively.

Let $D = \{\phi_j \in F(\Gamma^*,O) \mid j \in J\}$ be an indexed set of functions. A pair $(A,\zeta)$ is said to be an automaton realization of $D$ if $A = (Q,\Gamma,O,\delta,\lambda)$, $\zeta : J \rightarrow Q$ and

$$\lambda(\zeta(j),w) = \phi_j(w), \forall w \in \Gamma^*, j \in J$$

An automaton $A$ is said to be a realization of $D$ if there exists a $\zeta : J \rightarrow Q$ such that $(A,\zeta)$ is a realization of $D$.

Let $(A,\zeta)$ and $(A',\zeta')$ be two automaton realizations. Assume that

$$A = (Q,\Gamma,O,\delta,\lambda)$$
Theorem 2. A map \( \phi : Q \to Q' \) is said to be an automaton morphism from \((A, \zeta)\) to \((A', \zeta')\), denoted by \( \phi : (A, \zeta) \to (A', \zeta') \) if \( \phi(\delta(q, \gamma)) = \delta'(\phi(q), \gamma), \forall q \in Q, \gamma \in \Gamma, \lambda(q) = \lambda'(\phi(q)), \forall q \in Q, \phi(\zeta(j)) = \zeta'(j), j \in J \). It is easy to see that composition of two automaton morphisms is again an automaton morphism. The automaton morphism \( \phi \) is called injective (surjective) if the map \( \phi \) is injective (surjective). If \( \phi \) is a bijection, then \( \phi^{-1} : (A', \zeta') \to (A, \zeta) \) is an automaton morphism too. An automaton realization \((A, \zeta)\) of \( D \) is called minimal if for each automaton realization \((A', \zeta')\) of \( D \) \( \text{card}(A) \leq \text{card}(A') \). Let \( \phi : \Gamma^* \to O \). For every \( w \in \Gamma^* \) define \( w \circ \phi : \Gamma^* \to O \) - the left shift of \( \phi \) by \( w \) as \( w \circ \phi(v) = \phi(wv) \). For \( D = \{ \phi_j \in F(\Gamma^*, O) \mid j \in J \} \) define the set \( W_D = \{ w \circ \phi_j : \Gamma^* \to O \mid w \in \Gamma^*, j \in J \} \).

An automaton \( A = (Q, \Gamma, O, \delta, \lambda) \) is called reachable from \( Q_0 \subseteq Q \), if

\[
\forall q_1 \in Q : \exists w \in \Gamma^*, q_0 \in Q_0 : q = \delta(q_0, w)
\]

A realization \((A, \zeta)\) is called reachable if \( A \) is reachable from \( \text{Im} \zeta \). A realization \((A, \zeta)\) is called observable or reduced, if

\[
\forall q_1, q_2 \in Q : [\forall w \in \Gamma^* : \lambda(q_1, w) = \lambda(q_2, w)] \implies q_1 = q_2
\]

Below we will review the main results on realization theory of Moore-automata. The results are classical, in fact, they are the oldest results on realization theory. For more on the topic see [5, 6].

Let \( D = \{ \phi_j : \Gamma^* \to O \mid j \in J \} \) be an indexed set of input-output maps. Let \( A = (Q, \Gamma, O, \delta, \lambda) \) a Moore automaton, \( \zeta : J \to Q \) and assume that \((A, \zeta)\) is a realization of \( D \). Define the realization \((A_r, \zeta_r)\) by \( A_r = (Q_r, \Gamma, O, \delta_r, \lambda_r) \), \( Q_r = \{ q \in Q \mid \exists j \in J, w \in \Gamma^* : \delta_r(q, w) = q \} \), \( \delta_r(q, \gamma) = \delta(q, \gamma), q \in Q_r, \gamma \in \Gamma, \zeta_r(j) = \zeta(j) \). It is easy to see that \((A_r, \zeta_r)\) is well-defined, it is reachable and \( \text{card}(A_r) \leq \text{card}(A) \). Moreover, \( \text{card}(A_r) < \text{card}(A) \) if and only if \( A \) is not reachable. Thus, all minimal realizations are reachable. Indeed, if \((A, \zeta)\) is a minimal realization of \( D \) and it is not reachable, then \((A_r, \zeta_r)\) is a realization of \( D \) such that \( \text{card}(A_r) < \text{card}(A) \). But this contradicts minimality of \((A, \zeta)\). The following result is a simple reformulation of the well-known properties of realizations by automaton. For references see [5, 6].

**Theorem 1.** Let \( D = \{ \phi_j \in F(\Gamma^*, O) \mid j \in J \} \). \( D \) has a realization by a finite Moore-automaton if and only if \( W_D \) is finite. In this case a realization of \( D \) is given by \((A_{can}, \zeta_{can})\) where \( A_{can} = (W_D, \Gamma, O, L, T) \), \( \zeta_{can}(j) = \phi_j \) and

\[
L(\phi, \gamma) = \gamma \circ \phi, T(\phi) = \phi(\epsilon), \phi \in W_D, \gamma \in \Gamma
\]

The realization \((A_{can}, \zeta_{can})\) is reachable and observable.

The realization \((A_{can}, \zeta_{can})\) is called the free realization. The following theorem gives equivalent conditions for minimality of a realization.

**Theorem 2.** Let \((A, \zeta)\) be a finite Moore-automaton realization of \( D = \{ \phi_j \in F(\Gamma^*, O) \mid j \in J \} \). The following are equivalent:

\[
\begin{align*}
&\text{A realization } (A, \zeta) \text{ is minimal,} \\
&\text{and } A' = (Q', \Gamma, O, \delta', \lambda')
\end{align*}
\]
(i) \((A, \zeta)\) is minimal,

(ii) \((A, \zeta)\) is reachable and observable,

(iii) \(\text{card}(A) = \text{card}(W_D)\),

(iv) For each reachable realization \((A', \zeta')\) of \(D\) there exists a surjective automaton morphism \(T : (A', \zeta') \rightarrow (A, \zeta)\). In particular, all minimal realizations of \(D\) are isomorphic.

The realization \((A_{can}, \zeta_{can})\) is minimal.

For \(\phi : \Gamma^* \rightarrow O\) define \(\phi_N = \phi|_{\{w \in \Gamma^* | |w| < N\}}\). That is, \(\phi_N : \{w \in \Gamma^* | |w| < N\} \rightarrow O, \phi_N(w) = \phi(w)\) for all \(w \in \Gamma^*, |w| < N\).

Let \(D = \{\phi_j \in F(\Gamma^*, O) | j \in J\}\). Let \(A = (Q, \Gamma, O, \delta, \lambda), \zeta : J \rightarrow Q\).

The pair \((A, \zeta)\) is said to be \(N\)-partial realization of \(D\) if \(\forall w \in \Gamma^*, |w| < N : \lambda(\zeta(j), w) = \phi_j(w)\). For each \(N, M > 0\) define

\[W_{D,N,M} = \{(w \circ \phi_j)_M | j \in J, w \in \Gamma^*, |w| < N\}\]

Define the sets \(W_{D,N,N} = \{\psi_N | \psi \in W_D\}\) and \(W_{D,N,\cdot} = \{w \circ \phi_j | j \in J, w \in \Gamma^*, |w| < N\}\). Define the map \(\eta_N : W_D \rightarrow W_{D,N,N}\) by \(\eta_N(\psi) = \psi_N\). The following holds.

**Theorem 3 (Partial realization by automata).**

(i) If \((A, \zeta)\) is a realization of \(\Phi\) and \(\text{card}(A) \leq N\), then

\[\text{card}(W_{D,N,N}) = \text{card}(W_{D,N,N+1}) = \text{card}(W_{D,N+1,N}) = \text{card}(W_D)\]

(ii) If \(\text{card}(W_{D,N,N+1}) = \text{card}(W_{D,N,N}) = \text{card}(W_{D,N,N})\), then \((A_N, \zeta_N)\) is an \(N\)-partial realization of \(D\) where \(A_N = (W_{D,N,N}, \Gamma, O, \delta, \lambda)\) such that for each \(w \in \Gamma^*, |w| < N, j \in J, \delta((w \circ \phi_j)_N) = (w \circ \phi_j)_N\), \(\forall j \in W_{D,N,N} : \lambda(f) = f(e), \forall j \in J, \zeta(j) = \phi_j\).

(iii) If \(\text{card}(W_{D,N,N}) = \text{card}(W_D)\), then \(\text{card}(W_{D,N,N+1}) = \text{card}(W_{D,N+1,N}) = \text{card}(W_{D,N,N})\) and \((A_N, \zeta_N)\) is a minimal realization of \(D\). In particular, if \(D\) has a realization \((A, \zeta)\) such that \(N \geq \text{card}(A)\), then \((A_N, \zeta_N)\) is a minimal realization of \(D\).

It is easy to see that reachability and observability of a finite Moore-automaton realization can be checked by an algorithm, provided that we can decide whether two elements of the output space are equal. It is also easy to see that any finite Moore-automaton realization can be transformed to a reachable and/or observable finite Moore-automaton realization realizing the same family of input-output maps. Moreover, if \(W_{D,N,N}\) satisfies condition (iii) of Theorem 3, then a minimal realization of \(D\) can be computed from \(W_{D,N,N}\).

## 4 Formal Power Series

The section presents the necessary results on formal power series. For more on the classical theory of rational formal power series, see [2, 16]. The results of the current section are extensions of the classical ones. The material of the current section can be found in [10, 13, 12].
Let $X$ be a finite alphabet. A formal power series $S$ with coefficients in $\mathbb{R}^p$ is a map

$$S : X^* \rightarrow \mathbb{R}^p$$

We denote by $\mathbb{R}^p \ll X^* \gg$ the set of all formal power series with coefficients in $\mathbb{R}^p$. An indexed set of formal power series $\Psi = \{ S_j \in \mathbb{R}^p \ll X^* \gg | j \in J \}$ is called rational if there exists a vector space $X$ over $\mathbb{R}$, $\dim X < +\infty$, linear maps

$$C : X \rightarrow \mathbb{R}^p$$

and an indexed set $B = \{ B_j \in \mathbb{X} | j \in J \}$ of elements of $\mathbb{X}$ such that for all $\sigma_1, \ldots, \sigma_k \in X, k \geq 0$, $S_j(\sigma_1\sigma_2\cdots\sigma_k) = CA_{\sigma_k}A_{\sigma_{k-1}}\cdots A_{\sigma_1}B_j$.

The 4-tuple $R = (X, \{ A_\sigma \}_{\sigma \in \mathbb{X}}, B, C)$ is called a representation of $S$. The number $\dim X$ is called the dimension of the representation $R$ and it is denoted by $\dim R$. In the sequel the following short-hand notation will be used $A_w := A_{w_k}A_{w_{k-1}}\cdots A_{w_1}$ for $w = w_1 \cdots w_k$. $A_w$ is the identity map.

Notice that the representation $R$ can naturally be viewed as a Moore-automaton with the infinite state-space $\mathbb{X}$. In [13] the interpretation of representations as Moore-automata is described in more detail. A representation $R_{min}$ of $\Psi$ is called minimal if for each representation $R$ of $\Psi$ it holds that $\dim R_{min} \leq \dim R$.

It is easy to see that if $\Psi$ rational and $\Psi \subseteq \Psi'$, then $\Psi'$ is rational.

Define $w \circ S \in \mathbb{R}^p \ll X^* \gg$ - the left shift of $S$ by $w$ by

$$\forall v \in X^* : w \circ S(v) = S(wv)$$

The following statements are generalizations of the results on rational power series from [2]. Let $\Psi = \{ S_j \in \mathbb{R}^p \ll X^* \gg | j \in J \}$. Define $W_\Psi$ by

$$W_\Psi = \text{Span}\{ w \circ S_j \in \mathbb{R}^p \ll X^* \gg | j \in J, w \in X^* \}$$

Define the Hankel-matrix $H_\Psi$ of $\Psi$ as $H_\Psi \in \mathbb{R}^{(X^* \times I) \times (X^* \times J)}$, $I = \{ 1, 2, \ldots, p \}$ and

$$(H_\Psi)_{(u,i),(v,j)} = (S_j)_{i}(vu)$$

Notice that $\dim W_\Psi = \text{rank } H_\Psi$.

**Theorem 4.** Let $\Psi = \{ S_j \in \mathbb{R}^p \ll X^* \gg | j \in J \}$. The following are equivalent.

(i) $\Psi$ is rational.

(ii) $\dim W_\Psi = \text{rank } H_\Psi < +\infty$,

(iii) The tuple $R_\Psi = (W_\Psi, \{ A_\sigma \}_{\sigma \in \mathbb{X}}, B, C)$, where $A_\sigma : W_\Psi \rightarrow W_\Psi$, $A_\sigma(T) = \sigma \circ T$, $B = \{ B_j \in W_\Psi | j \in J \}$, $B_j = S_j$ for each $j \in J$, $C : W_\Psi \rightarrow \mathbb{R}^p$, $C(T) = T(\epsilon)$, defines a representation of $\Psi$.

The representation $R_\Psi$ is called free. Since the linear space spanned by the column vectors of $H_\Psi$ and the space $W_\Psi$ are isomorphic, one can construct a representation of $\Psi$ over the space of column vectors of $H_\Psi$ in a way similar to the construction of $R_\Psi$. 
Let $R = (\mathcal{X}, \{A_x\}_{x \in \mathcal{X}}, B, C)$ be a representation of $\Psi$. Define the subspaces $W_R$ and $O_R$ of $\mathcal{X}$ by

$$W_R = \text{Span}\{A_w B_j \mid w \in X^*, j \in J\} \quad \text{and} \quad O_R = \bigcap_{w \in X^*} \ker CA_w$$

A representation $R$ is called observable, if $O_R = \{0\}$. A representation $R$ is called reachable, if $\dim R = \dim W_R$.

It is easy to see that if $n = \dim \mathcal{X}$, then

$$O_R = \bigcap_{w \in X^*, |w| \leq n} \ker CA_w \quad \text{and} \quad W_R = \text{Span}\{A_w B_j \mid j \in J, |w| \leq n\}$$

That is, if $J$ is a finite set, then observability and reachability of representations can be checked by checking whether certain finite matrices are of full rank. Moreover, if $R$ is a representation of $\Psi$, then $R$ can be transformed to a reachable representation of $\Psi$:

$$R_r = (W_R, \{A_x\}_{x \in \mathcal{X}}, B, C)$$

It can also be transformed to an observable representation of $\Psi$:

$$R_o = (\mathcal{X}/O_R, \{A_x^{obs}\}_{x \in \mathcal{X}}, B^{obs}, C^{obs})$$

where $C^{obs}(x + O_R) = Cx$, $B^{obs}_j = B_j + O_R$, $A_x^{obs}(x + O_R) = A_x x + O_R$. The constructions above are computable from $R$ if $J$ is finite.

Let $R = (\mathcal{X}, \{A_x\}_{x \in \mathcal{X}}, B, C)$ and $\tilde{R} = (\tilde{\mathcal{X}}, \{\tilde{A}_x\}_{x \in \mathcal{X}}, \tilde{B}, \tilde{C})$ be two representations of $\Psi$. Then a linear map $T : \tilde{\mathcal{X}} \to \mathcal{X}$ is called a representation morphism from $\tilde{R}$ to $R$, denoted by $T : \tilde{R} \to R$, if

$$T \tilde{A}_x = A_x T, (x \in \mathcal{X}) \quad T \tilde{B}_j = B_j, (j \in J), \quad \tilde{C} = CT$$

The representation morphism $T$ is said to be injective (surjective), if it is an injective (surjective) linear map. A representation isomorphism is simply a bijective representation morphism. Two representations are said to be isomorphic, if there exists a representation isomorphism between them.

Let $R = (\mathcal{X}, \{A_x\}_{x \in \mathcal{X}}, B, C)$ be a representation and let $W \subseteq \mathcal{X}$ be a linear subspace of $\mathcal{X}$. $R$ is said to be $W$-observable, if $W \cap O_R = \{0\}$. It is clear that if $R$ is observable, then $R$ is $W$-observable for any subspace $W$. It is also easy to see that if $R$ is $W$-observable and $T : R \to R'$ is a representation morphism then $T|_W$ is an injective linear map.

**Theorem 5 (Minimal representation).** Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$. The following are equivalent.

(i) $R_{min}$ is a minimal representation of $\Psi$,

(ii) $R_{min}$ is reachable and observable,

(iii) rank $H_\Psi = \dim W_\Psi = \dim R_{min}$,

(iv) If $R$ is a reachable representation of $\Psi$, then there exists a surjective representation morphism $T : R \to R_{min}$. 


In particular, if $R$ is a minimal representation, then $T$ is a representation isomorphism.

Using the theorem above it is easy to check that the free representation $R_{\Psi}$ is minimal. One can also give a procedure, similar to reachability and observability reduction for linear systems, such that the procedure transforms any representation of $\Psi$ to a minimal representation of $\Psi$. If $R = (\mathcal{X}, \{A_{\sigma}\}_{\sigma \in \Sigma}, B, C)$ is a representation of $\Psi$, then for any vector space isomorphism $T : \mathcal{X} \rightarrow \mathbb{R}^n$, $n = \dim R$, the tuple $R' = (\mathbb{R}^n, \{TA_{\sigma}T^{-1}\}_{\sigma \in \Sigma}, TB, CT^{-1})$ is also a representation of $\Psi$. It is easy to see that $R$ is minimal if and only if $R'$ is minimal.

From now on, we will silently assume that $\mathcal{X} = \mathbb{R}^n$ holds for any representation considered.

For each $S \in \mathbb{R}^p \ll X^* \gg$ define $S_N = S\{w \in X^* \mid |w| < N\}$. Let $\Psi = \{S_j \in \mathbb{R}^p \ll X^* \gg \mid j \in J\}$ and $R = (\mathcal{X}, \{A_{\sigma}\}_{\sigma \in \Sigma}, C, B), B = \{B_j \in \mathcal{X} \mid j \in J\}$. The representation $R$ is said to be an $N$-partial representation of $\Psi$ if for each $j \in J$, $w \in X^*, |w| < N$ it holds that $S_j(w) = CA_wB_j$. Let $H_{\Psi, N, M} \in \mathbb{R}^{JM \times JN}, J_M = \{(u, i) \mid v \in X^*, |v| < M, i = 1, \ldots, p\}, J_N = \{(u, j) \mid j \in J, u \in X^*, |u| < N\}$ and $(H_{\Psi, N, M}((u, i), (u, j))) = ((S_j(wv)))$. Notice that $H_{\Psi, N, M}$ is a finite matrix, if $J$ is finite. Define $W_{\Psi, N, M} = \{(w \circ S_j)_{|M} \mid w \in X^*, |w| < N, j \in J\}$. Notice that rank $H_{\Psi, N, M} = \dim W_{\Psi, N, M}$.

**Theorem 6 (Partial representation).** (i) If $R$ is a representation of $\Psi$, $\dim R \leq N$, then

$$\text{rank } H_{\Psi} = \text{rank } H_{\Psi, N, N}$$

(ii) If rank $H_{\Psi, N, N} = \text{rank } H_{\Psi, N, N+1} = \text{rank } H_{\Psi, N+1, N}$, then there exists an $N$-representation

$$R_N = (W_{\Psi, N, N}, \{A_{\sigma}\}_{\sigma \in \Sigma}, C, B)$$

of $\Psi$, such that $A_{\sigma}((w \circ S_j)_{N}) = (w \circ S_j)_{N}, C(T) = T(\epsilon), B_j = (S_j)_{|N}, j \in J$.

(iii) If $\Psi$ has a representation $R$ such that $N \geq \dim R$, then $R_N$ is a minimal representation of $\Psi$.

The theorem above implies that if $J$ is finite and we know that $\Psi$ has a representation of dimension at most $N$, then a minimal representation of $\Psi$ can be computed from finite data.

## 5 Hybrid Formal Power Series

The section introduces the concept of hybrid power series and hybrid power series representation. This section contains the main contribution of the paper. Subsection 5.1 contains the definition and basic properties of hybrid formal power series and hybrid representations. Subsection 5.2 discusses the problem of existence of hybrid representations. It gives necessary and sufficient conditions for a family of hybrid formal power series to admit a hybrid representation. Subsection 5.3 characterises minimal hybrid representations. Throughout the section the notation of Section 4 will be used.

### 5.1 Definitions and Basic Properties

Let $X$ be an alphabet, i.e. a finite set and let $O$ be an arbitrary finite set. Assume that $X = X_1 \cup X_2$ such that $X_1 \cap X_2 = \emptyset$. We allow $X_1$ or $X_2$ to be
the empty set. Let $J$ be any set of the following form.

\[ J = J_1 \cup (J_1 \times J_2) \]

\[ J_2 \text{ is a finite set, } J_2 \cap J_1 = \emptyset \]  

(1)

Sets with the property (1) above will be called \textit{hybrid power series index sets}. Notice that we allow $J_2$ to be the empty set.

A \textit{hybrid formal power series} over $X_1, X_2$ with coefficients in $\mathbb{R}^p \times O$ is a pair

\[ S = (S_C, S_D) \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O) \]

That is, a hybrid formal power series $S$ is a pair of functions. The first component of the pair is a map $S_C : X^* \to \mathbb{R}^p$, the second component is a map $S_D : X_2^* \to O$. We will denote the set of all hybrid formal power series over $X_1, X_2$ with coefficients in $\mathbb{R}^p \times O$ by $\mathbb{R}^p \ll X^* \gg \times F(X_2^*, O)$. If the space of coefficients and the alphabets $X_1, X_2$ are clear from the context we will simply speak of hybrid formal power series. If $S \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O)$ is a hybrid formal power series, then define the formal power series $S_C \in \mathbb{R}^p \ll X^* \gg$ and the map $S_D : X_2^* \to O$ in such a way that $S = (S_C, S_D)$. That is, $S_D$ denotes the discrete valued ($O$ valued) component of $S$ and $S_C$ denotes the continuous ($\mathbb{R}^p$) valued component of $S$.

Assume that $S$ is a hybrid formal power series index set. Let $\Omega = \{ Z_j \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O) \mid j \in J \}$ be an indexed set of hybrid formal power series indexed by $J$ such that

\[ \forall k \in J_1, j \in J_2 : (Z_{k,j})_D = (Z_k)_D \text{ and } (Z_{k,j})_C(w) = 0, \forall w \in X_2^* \]  

(2)

Indexed sets of hybrid formal power series with the property (2) above will be called \textit{well-posed indexed sets of hybrid power series}. The intuition behind the definition of well-posed indexed sets of hybrid power series is the following. We can think of the indexed set $\Omega$ as an encoding of the indexed set $\Psi = \{ f_j \mid j \in J_1 \}$, where $f_j : X \ni w \mapsto ((Z_{j})_C(w), (Z_{j})_D(v), ((Z_{j,k})_C)(w))_{k \in J_2}$, where $v = \gamma_1 \cdots \gamma_k \in X_2^*$ and $w$ is assumed to be of the form $w = z_1 \gamma_1 z_2 \cdots \gamma_k z_{k+1}, z_1, \ldots, z_{k+1} \in X_1^*$, $\gamma_1, \ldots, \gamma_k \in X_2$. The indexed set $\Psi$ is supposed to contain input-output maps of a system which is an interconnection of a special form of a finite Moore-automaton and formal power series representations. The requirement $(Z_{j,k})_C(w) = 0$ for all $w \in X_2^*$ reflects the special structure of this interconnection. The motivation of the definition of a well-posed indexed set of hybrid power series should become clear to the reader after seeing the definition of a hybrid power series representation. A hybrid formal power series representation defines exactly an interconnection of a Moore-automaton and formal power series representations such that the input-output maps of the interconnection can be encoded by a well-defined indexed set of hybrid formal power series.

In the sequel, we will mostly work with well-posed indexed sets of hybrid formal power series. In the rest of the paper, unless stated otherwise, we will always mean a well posed indexed set of hybrid formal power series whenever we speak of indexed sets of hybrid formal power series.

\textbf{Definition 1.} A \textit{hybrid representation} (abbreviated by HR) over $J$ is a tuple

\[ HR = (A, \mathcal{Y}, \{ X_q, \{ A_{q,z}, B_{q,z,j_2} \}_{j_2 \in J_2, z \in X_1}, C_q, \{ M_{k(q,y),y} \}_{y \in X_2} \}_{q \in Q}, J, \mu) \]

where
$\mathcal{A} = (Q, X_2, O, \delta, \lambda)$ is a Moore-automaton

$X_q$ is a finite-dimensional vector space for all $q \in Q$. Without loss of generality we can assume that $X_q = \mathbb{R}^{n_q}$ for some $n_q > 0$.

$\mathcal{Y}$ is a finite-dimensional vector space and $\mathcal{Y} = \mathbb{R}^p$ for some $p \in \mathbb{N}, p > 0$.

$M_{q_1, x, q_2} : X_{q_2} \rightarrow X_{q_1}$ is a linear map, for each $q_1, q_2 \in Q$, $x \in X_2$ such that $\delta(q_2, x) = q_2$.

$A_{q, x} : X_q \rightarrow X_q$ is a linear map for each $x \in X_1$ and $q \in Q$.

$C_q : X_q \rightarrow \mathcal{Y}$ is a linear map for each $q \in Q$.

For each $q \in Q$, $j \in J_2$, $x \in X_1$, the vector $B_{q, x, j}$ belongs to $X_q$, i.e. $B_{q, x, j} \in X_q$.

Define $\mu : J_1 \rightarrow \bigcup_{q \in Q} \{ q \} \times X_q$ is a map

Define $\mu_D : J_1 \rightarrow Q$ and $\mu_C : J_1 \rightarrow \bigcup_{q \in Q} X_q$ by

$$\forall j \in J_1 : \mu(j) = (q, x) \iff \mu_D(j) = q \text{ and } \mu_C(j) = x$$

If $J_2 = \emptyset$, then we will use the following short-hand notation for the hybrid representation $HR$

$$(\mathcal{A}, (X_q, \{ A_{q, x} \}_{x \in X_1}, C_q)_{q \in Q}, \{ M_{s(q, y), y, q} | q \in Q, y \in X_2 \}, J, \mu)$$

In fact, a hybrid representation can be viewed as a some sort of cascade interconnection of a Moore-automaton and formal power series representations. Recall from Section 4 that a formal powers series representation can be thought of as a Moore-automaton, state-space of which is a vector space (thus, not necessarily finite). One could define a suitable notion of cascade interconnection for Moore-automata, see for example [5] and view a hybrid representation as an interconnection of a finite Moore-automaton with a number of Moore-automata which are in fact formal power series representations.

A hybrid representation can be itself viewed as a Moore-automaton. Before we can explain how to view a hybrid representation as a Moore-automata, we will need some additional definitions and notation.

Define the set

$$\hat{O} = \prod_{j \in J_2} \mathbb{R}^p \lhd X^* \rhd$$

An element of the set $\hat{O}$ is a tuple $(S_j)_{j \in J_2}$ such that $S_j \in \mathbb{R}^p \lhd X^* \rhd$ for all $j \in J_2$. If $J_2 = \emptyset$ then $\hat{O}$ will be viewed as the singleton set $\{ \emptyset \}$.

Denote by $\mathcal{H}_{HR}$ the set $\mathcal{H}_{HR} = \bigcup_{q \in Q} \{ q \} \times X_q$. Define the maps $\Pi_Q : \mathcal{H}_{HR} \ni (q, x) \mapsto q \in Q$ and $\Pi_X : \mathcal{H}_{HR} \ni (q, x) \mapsto x \in \bigcup_{q \in Q} X_q$.

Consider any $w \in X^*$. It is easy to see that $w$ can be represented as $w = x_1 y_1 x_2 y_2 \cdots x_k y_k x_{k+1}$, for some $x_1, x_2, \ldots, x_k \in X_1^*$, $y_1, y_2, \ldots, y_k \in X_2$ and $k \geq 0$. It is easy to see that the representation above is unique. Such a representation can be easily obtained by grouping together those letters of $w$ which belong to $X_1$.

The reader who wishes to see a formal proof, will find one below. The proof goes by induction. If $|w| = 1$, then $w = w_1$ and either $w_1 \in X_1$ or...
$w_1 \in X_2$. If $w_1 \in X_1$ then set $k = 0$ and $x_1 = w_1$. If $w_1 \in X_2$, then set $k = 1$, $y_1 = w_1$ and $x_1 = x_2 = \epsilon$. In both cases $w = x_1 y_1 \cdots y_k x_{k+1}$. Assume that a representation of the above form exists for all words $w \in X^*$, $|w| \leq n$. Assume that $w = w_1 \cdots w_{n+1}$, $w_1, \ldots, w_{n+1} \in X$. For each $i = 1, \ldots, n+1$ either $w_i \in X_1$ or $w_i \in X_2$. Assume that $w_1, w_2, \ldots, w_i \in X_1$ and $w_{i+1} \in X_2$. Let $x_1 = w_1 \cdots w_j \in X_1^*$ and $y_1 = w_{j+1} \in X_2$. If $w_1 \in X_2$ then $j = 0$ and $x_1 = \epsilon$. Consider the representation of $v = w_{j+2} \cdots w_{n+1}$, i.e. assume that $v = x_2 y_2 \cdots y_k x_{k+1}, x_2, \ldots, x_k, y_k \in X_2$. Such a representation of $v$ exists by the induction hypothesis. Then $w = x_1 y_1 v = x_1 y_1 x_2 \cdots y_k x_{k+1}$, that is, $x_1, \ldots, x_{k+1} \in X_1^*, y_1, \ldots, y_k \in X_2$.

For each $q \in Q$, $w = x_1 \cdots x_k \in X_1^*$, $x_1, \ldots, x_k \in X_1$ denote by $A_{q,w}$ the composition of linear maps $A_{q,x_1} A_{q,x_{k-1}} \cdots A_{q,x_1}$. If $k = 0$, i.e. $w = \epsilon$ then let $A_{q,w} = A_{q,\epsilon}$. For each $q \in Q$, $w = x_1 \cdots x_k \in X_1^*$, define the map $\xi_{HR}: H_{HR} \times X^* \rightarrow H_{HR}$ by

$$\xi_{HR}(q, w) = \left( \delta(q, w_1, \cdots, w_k), A_{q_1, z_{k+1}} M_{q_k, w_k, q_{k-1}} A_{q_{k-1}, z_k} \cdots A_{q_1, z_2} M_{q_1, w_1, q_0} A_{q_0, z_1} x \right)$$

for all $z_1, \ldots, z_{k+1} \in X_1^*$, $w_1, \ldots, w_k \in X_2$, $k \geq 0$, where $q_l = \delta(q, w_1, \cdots, w_l)$ for all $i = 0, \ldots, k$ (i.e. $q_0 = q$).

For each $q \in Q$, $j \in J_2$ define the power series $T_{q,j} = \mathbb{R}^p \ll X^*$ as follows. Recall that each $w \in X^*$ can be uniquely written as $w = x_1 y_1 x_2 \cdots y_k x_{k+1}$, for some $y_1, \ldots, y_k \in X_2$, $x_1, \ldots, x_{k+1} \in X_1$ and $k \geq 0$. Then for each $w \in X^*$ define $T_{q,j}(w)$ as

$$T_{q,j}(w) = T_{q,j}(x_1 y_1 \cdots x_k y_k x_{k+1}) = C_{q_l} A_{q_1, y_1, x_{k+1}} M_{q_k, y_k, q_{k-1}} A_{q_{k-1}, z_k} \cdots M_{q_1, y_1, q_0} A_{q_0, z_1} x_{k+1} B_{q_{k+1}, s_{l+1}}$$

where $1 \leq l \leq k + 1$, $x_1 = x_2 = \cdots = x_{l-1} = \epsilon$, $x_l = s_l z_l$, $s_l \in X_1$, $z_l \in X_1^*$, $q_l = \delta(q, y_1, \cdots, y_l)$ for all $i = 0, \ldots, k$.

The tuple $(T_{q,j})_{j \in J_2} \in \mathcal{O}$ will serve as the output of the hybrid representation $H_R$. Define the map $v_{HR}: H_{HR} \times X^* \rightarrow \mathbb{R}^p \times \mathcal{O} \times \mathcal{O}$ as follows

$$\forall w \in X^* : v_{HR}(q, x) = (C_s z, \lambda(s), (T_{q,j})_{j \in J_2})$$

where $(s, z) = \xi_{HR}(q, x)$.

The map $\xi_{HR}$ plays the role of state-trajectories and $v_{HR}$ plays the role of output-trajectories of the automaton associated with the hybrid representation $H_R$.

Now we are in position to explain the analogy between hybrid representations and Moore-automata. A hybrid representation $H_R$ can be viewed as an infinite-state Moore-automata, which is defined as follows. Its state space is the set $H_{HR}$. Each state is a pair $(q, x)$, consisting of a discrete component $q$ and a continuous component $x \in X_q$. The input alphabet of a hybrid representation viewed as a Moore-automaton is $X$. The output alphabet is the set $\mathbb{R}^p \times \mathcal{O} \times \mathcal{O}$. The state-space evolution of a hybrid representation can be viewed as follows. If the hybrid representation receives a symbol $z \in X_1$, then the state changes as follows. If the current state is of the form $(q, x) \in \{q\} \times X_q$ then the current state changes to $(q, A_{q,z} x)$. If the hybrid representation receives a symbol $y \in X_2$ then the state of the hybrid representation changes as follows. If the current state is of the form $(q, x) \in \{q\} \times X_q$ then the current state changes to $(\delta(q, y), M_{\delta(q,y), y,q} x) \in \{\delta(q, y)\} \times X_{\delta(q,y)}$. If the current state is
of the form \((q, x) \in \{q\} \times X_q\), then the output of the hybrid representation is 
\((C_q x, \lambda(q), (T_{q,j})_{j \in J}^2)\). The tuple \((T_{q,j})_{j \in J}\) can be thought as an analog of the 
impulse response for linear systems. The map \(\mu\) can be thought of as a way 
to define the set of initial states of the Moore-automaton interpretation of the 
hybrid representation. Namely, the set of initial states is made up by the states 
\(\mu(j) \in \mathcal{H}_{HR}, \ j \in J\).

We will not use the interpretation of a hybrid power series representation as a 
Moore-automaton presented above to prove mathematical properties of hybrid 
representations. However, we will frequently refer to this interpretation in order 
to give an intuitive description of results and concepts.

We define the dimension of the hybrid representation \(HR\) as the pair 
\((\text{card}(Q), \sum_{q \in \mathcal{Q}} \dim X_q)\)
and it is denoted by \(\dim HR\). We will use the following partial order relation 
on \(\mathbb{N} \times \mathbb{N}\). We will say that \((p, q) \in \mathbb{N}\) is smaller than or equal \((r, s) \in \mathbb{N}\) if \(p \leq r\) 
and \(q \leq s\). We will denote the fact that \((p, q)\) is smaller than or equal \((r, s)\) by 
\((p, q) \leq (r, s)\). Note the the order relation \(\leq\) in \(\mathbb{N} \times \mathbb{N}\) is indeed a partial order, 
it is not possible to compare all elements of \(\mathbb{N} \times \mathbb{N}\).

Consider an indexed set of hybrid formal power series \(\Omega = \{X_j \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O) \mid j \in J\} \cup J_1 \times J_2\). The hybrid representation 
\(HR\) is said to be a hybrid representation of \(\Omega\) if for all \(w = x_1 y_1 \cdots x_k y_k x_{k+1} \in X^*\), \(x_i \in X_1^*, y_j \in X_2, i = 1, 2, \ldots, k + 1, j = 1, 2, \ldots, k, k \geq 0\) the following 
holds
\begin{align*}
\forall j \in J_1 : \quad & (Z_j)_C(w) = C_{q_k} A_{q_k, x_{k+1}} M_{q_k, y_k, q_{k-1}} A_{q_{k-1}, x_k} \cdots M_{q_1, y_1, q_0} A_{q_0, x_1} \mu_C(j) \\
\forall j \in J_1 : \quad & (Z_j)_D(y_1 \cdots y_k) = \lambda(\mu_D(j), y_1 \cdots y_k) \\
\forall (j_1, j_2) \in J_1 \times J_2 : \quad & (Z_{j_1, j_2})_C(w) = C_{q_k} A_{q_k, x_{k+1}} M_{q_k, y_k, q_{k-1}} A_{q_{k-1}, x_k} \cdots \\
& \cdots M_{q_1, y_1, q_1} A_{q_1, x_1} \mu_{C}(Z_{j_1, j_2})(w) = 0
\end{align*}
where \(q_0 = \mu_D(j), q_l = \delta(q_0, y_1 \cdots y_l), 1 \leq l \leq k\). One can think of \((Z_j)_C\) as 
continuous output, \((Z_j)_D\) as discrete-output and \((Z_{j_1, j_2})_C\) as continuous output 
corresponding to the impulse response. This is of course only an analogy, there 
is no formal correspondence between the objects mentioned above. An indexed 
set of hybrid formal power series is called rational if it has a hybrid representa-
tion. Note that the framework above resembles very much the concept of 
rational representations described in [15]. In fact, when \(Q = \{q\}\) is a singleton 
set, the notion of hybrid representation and the notion of rational representa-
tion coincide. We say that the hybrid representation \(HR\) is a minimal hybrid 
representation of \(\Omega\) if \(HR\) is a hybrid representation of \(\Omega\) and for any hybrid 
representation \(\tilde{HR}\) of \(\Omega\)
\[
\dim HR \leq \dim \tilde{HR}
\]
Recall the interpretation of a hybrid representation as a Moore-automaton. Then the statement that $HR$ is a hybrid representation of $\Omega$ simply says that for each $j_1 \in J_1$ the Moore-automaton interpretation of the hybrid representation $HR$ realizes the map:

$$T_{j_1} : X^* \ni w \mapsto ((Z_{j_1})_C(w), (Z_{j_1})_D(\Pi X_2(w)), ((Z_{j_1,j_2})_C(w)))_{j_2 \in J_2}$$

from the initial states $\mu(j_1)$. Here $\Pi X_2 : X^* \rightarrow X_2^*$ is a map which erases all the letters not in $X_2$, i.e., $\Pi X_2(x_1 y_1 \cdots x_k y_k x_{k+1}) = y_1 \cdots y_k$ for each $x_1, \ldots, x_{k+1} \in X_1^+, y_1, \ldots, y_k \in X_2$, $k \geq 0$.

Thus $HR$ is a representation of $\Omega$ if and only if

$$\forall j \in J_1, \forall w \in X^* : ((Z_j)_C(w), (Z_j)_D(\Pi X_2(w)), (Z_{j,j_2}(w)))_{j_2 \in J_2} = v_HR(\mu(j), w) \tag{4}$$

Let $HR = (A, \gamma, (X_q, \{A_{q,z}, B_{q,z,j_2}\}_{j_2 \in J_2, z \in X_1}, C_q, \{M_{\gamma(q,y),q} \}_{q \in Q}, J, \mu)$ be a hybrid representation. Let

$$HR' = (A', \gamma', (X_q', \{A'_{q,z}, B'_{q,z,j_2}\}_{j_2 \in J_2, z \in X_1}, C_q', \{M'_{\gamma'(q,y),q} \}_{q \in Q'}, J, \mu')$$

be another hybrid representation. A pair $T = (T_D, T_C)$ is called a $HR$-morphism (hybrid representation morphism) from $HR$ to $HR'$ denoted by $T : HR \rightarrow HR'$ if $T_D : (A, \mu_D) \rightarrow (A', \mu_D')$ is an automaton realization morphism, $T_C : \bigoplus_{q \in Q} X_q \rightarrow \bigoplus_{q' \in Q'} X'_q$ is a linear map such that

$$T_C(X_q) \subseteq X'_{T_D(q)} \text{ for all } q \in Q,$$

$$T_CM_{q_1,q_2} = M'_{T_D(q_1),x,T_D(q_2)} T_C \text{ for all } q_1, q_2 \in Q, x \in X_2 \text{ such that } \delta(q_2, x) = q_1,$$

$$T_C A_{q,z} = A'_{T_D(q),z} C(T) \text{ for all } q \in Q, z \in X_1,$$

For all $q \in Q$, $j \in J_2$, $z \in X_1$, $T_C B_{q,z,j} = B'_{T_D(q),z,j}$

$$C_q = C'_{T_D(q)} T_C \text{ for each } q \in Q,$$

$$T_C \mu_C(j) = \mu'_C(j) \text{ for all } j \in J_1$$

It is easy to see that the pair $T = (T_D, T_C)$ defines a map

$$\phi(T) : \mathcal{H}_{HR} \ni (q, x) \rightarrow (T_D(q), T_C(x)) \in \mathcal{H}_{HR'}$$

In fact, if we use the interpretation of hybrid representations as Moore-automata, then $\phi(T)$ defines a Moore-automaton morphism.

We will call $HR$ observable if for each $h_1, h_2 \in \mathcal{H}_{HR}$

$$\forall w \in X^* : v_{HR}(h_1, w) = v_{HR}(h_2, w) \implies h_1 = h_2$$

Define the set

$$\mathcal{H}_{0,HR} = \{(q, x) \mid (\exists j \in J_1 : \mu(j) = (q, x)) \text{ or } (q = \delta(\mu_D(j), v), x = B_{q,z,j}, \text{ for some } v \in X_2^*, z \in X_1, j \in J_2)\}$$
Define the set

\[ \text{Reach}(HR) = \{(q, x) | \exists w_1, \ldots, w_k \in X^*, a_1, \ldots, a_k \in \mathbb{R}, h_1, \ldots, h_k \in \mathcal{H}_{0,HR}, k \geq 0, \]
\[ x = \sum_{j=1}^{k} \alpha_j \Pi_X(\xi_{HR}(h_i, w_i)) \]
\[ \text{and } q = \Pi_Q(\xi_{HR}(h_i, w_i)), i = 1, \ldots, k \} \]

We will call \( HR \) reachable if \( \mathcal{H}_{HR} = \text{Reach}(HR) \).

Below we will give a reformulation of observability and reachability of hybrid representations. For the \( HR \) \( HR \) define the following spaces

\[ W_{HR} = \text{Span}( \{ A_{q_0, y_0, q_0, 1} A_{q_0, 1, y_k, q_0, 1} \cdots M_{q_0, y_1, q_0, 1} A_{q_0, 1, y_k, q_0, 1} \mu_C(j) \mid j \in J_1, x_1, \ldots, x_{k+1} \in X_{1}^*, y_1, \ldots, y_k \in X_2, \]
\[ q_0 = \mu_D(j), q_k = \delta(q_0, y_1 \cdots y_k), 1 \leq l \leq k, k \geq 0 \} \cup \]
\[ \cup \{ A_{q_0, y_0, q_0, 1} A_{q_0, 1, y_k, q_0, 1} \cdots M_{q_0, y_1, q_0, 1} A_{q_0, 1, y_k, q_0, 1} \mu_C(j) \mid j \in J_2, j \in J_1, x_1, \ldots, x_{k+1} \in X_{1}^*, y_1, \ldots, y_k \in X_2, \]
\[ q_0 = \mu_D(j), q_k = \delta(q_0, y_1 \cdots y_k), 1 \leq l \leq k, k \geq 0 \} \subseteq \bigoplus_{q \in Q} X_q \]

The following statement is an easy consequence of the definition.

**Proposition 1.** The hybrid representation \( HR \) is reachable, if and only if \((A, \mu_D)\) is reachable and \( W_{HR} = \bigoplus_{q \in Q} X_q \).

Again, if we look at the Moore-automaton interpretation of \( HR \), then \( W_{HR} \) is precisely the linear span of the continuous components of the states which belong to \( \bigcup_{q \in Q} \{ q \} \times X_q \) and can be reached from some initial state.

Below we will give a characterisation of observability of hybrid representations. For each \( q \in Q \), define

\[ O_{HR,q} = \bigcap_{q \in Q, w} O_{q,w} \]

where for all \( w = x_1 y_1 \cdots y_k x_{k+1} \in X^*, k \geq 0, x_1, \cdots, x_{k+1} \in X_{1}^*, y_1, \cdots, y_k \in X_2 \)

\[ O_{q,w} = \ker C_q A_{q_0, x_{k+1}, y_{k+1}} A_{q_0, 1, y_k, q_0, 1} A_{q_0, 1, y_k, q_0, 1} \cdots M_{q_0, y_1, q_0, 1} A_{q_0, 1} \]

where \( q = q_0 \in Q, \mu = \delta(q_0, y_1 \cdots y_k), 0 \leq l \leq k \). The space \( O_{HR,q} \) is analogous to the observability kernel of linear (bilinear) systems and plays a very similar role. Unfortunately, the spaces \( O_{HR,q} \) are not sufficient to characterise observability for hybrid representations. The following proposition characterises observability of hybrid representations.

**Proposition 2.** The hybrid representation \( HR \) is observable, if and only if the following two conditions hold
(i) For each \( q_1, q_2 \in Q \), if for all \( w \in X^*_2 \), \( j \in J_2 \)
\[
\lambda(q_1, w) = \lambda(q_2, w) \quad \text{and} \quad T_{q_1,j} = T_{q_2,j}
\]
then \( q_1 = q_2 \).

(ii) For each \( q \in Q \), \( O_{HR,q} = \{0\} \)

Notice that if \( J_2 = \emptyset \) then the first condition in the definition of observability
is equivalent to \( A \) being observable.

If we look at the Moore-automaton interpretation of hybrid representations,
then a hybrid representation is observable if and only if the Moore-automaton
interpretation of the hybrid representation is observable.

Next we will discuss certain elementary properties of hybrid representation
morphisms. Recall that any hybrid representation morphism \( T : HR \to HR' \)
induces a map \( \phi(T) : H_{HR} \to H_{HR'} \).

**Proposition 3.** A hybrid representation morphism \( T \) is a hybrid representation
isomorphism if and only if \( \phi(T) \) is a bijective map.

**Proposition 4.** Let \( HR_1 \) and \( HR_2 \) be two hybrid representations. Assume
that \( T : HR_1 \to HR_2 \) is a hybrid representation morphism. Then the following
holds.

- If \( T \) is injective, then \( \dim HR_1 \leq \dim HR_2 \).
- If \( T \) is surjective, then \( \dim HR_2 \leq \dim HR_1 \).
- If \( T \) is either injective or surjective and \( \dim HR_1 = \dim HR_2 \), then \( T \) is
  an hybrid representation isomorphism.

The following proposition gives an important system theoretic characterisation
of hybrid representation morphisms.

**Proposition 5.** Let \( HR_i, i = 1,2 \) be two hybrid representations and let \( T : HR_1 \to HR_2 \) be a hybrid representation morphism. Then the following holds.
\[
\phi(T)(\xi_{HR_1}(h,v)) = \xi_{HR_2}(\phi(T)(h),v) \quad \text{and} \quad \upsilon_{HR_1}(h,v) = \upsilon_{HR_2}(\phi(T)(h),v)
\]
for all \( h \in H_{HR_1}, v \in X^* \). If \( T \) is a hybrid representation isomorphism, then
\( HR_1 \) is reachable if and only if \( HR_2 \) is reachable and \( HR_1 \) is observable if and
only if \( HR_2 \) is observable.

**Corollary 1.** Let \( HR_1, HR_2 \) be hybrid representations and let \( T : HR_1 \to HR_2 \) be a hybrid representation morphism. Then \( HR_1 \) is a representation of an
indexed set of hybrid power series \( \Omega \) if and only if \( HR_2 \) is a representation of \( \Omega \).

### 5.2 Existence of Hybrid Representations

In this subsection we will give necessary and sufficient conditions for existence
of a hybrid representation for a family of hybrid formal power series. Recall that
hybrid representations can be viewed as an interconnection of Moore-automata
and rational representations. In the light of this remark it should not be surprising
that finding a hybrid representation for an indexed set of hybrid power series
can be reduced to finding a rational representation for an indexed set of formal power series and finding a finite Moore-automaton realization for an indexed set of discrete input-output maps.

We will proceed as follows. We will associate with each family of hybrid formal power series a family of classical formal power series and a family of discrete input-output maps. It turns out that there is a correspondence between rational representations of this family of formal power series and automaton realizations of the family of discrete input-output maps on the one hand and hybrid representations of the original family of hybrid formal power series on the other hand.

Let \( HR = (A, \mathcal{Y}, (X_q, \{ A_{q,z}, B_{q,z,j} \}_{j \in J_2, z \in X_1}, C_q, \{ M_{\delta(q,y), y,q} \}_{y \in X_2})_{q \in Q}, J, \mu) \) be a hybrid representation. Assume that \( A = (Q, \Gamma, O, \delta, \lambda), Q = \{ q_1, \ldots, q_N \} \) and \( \text{card}(J_2) = m \). Fix a basis \( \{ e_{q,j} \mid q \in Q, j \in J_2 \} \) in \( \mathbb{R}^m \). Define the representation associated with \( HR \) by

\[
R_{HR} = (\mathcal{X}, \{ M_z \}_{z \in \mathcal{X}}, \tilde{B}, \tilde{C})
\]

where

- \( \mathcal{X} = (\bigoplus_{q \in Q} X_q) \oplus \mathbb{R}^m \), if \( m > 0 \) and \( \mathcal{X} = \bigoplus_{q \in Q} X_q \) if \( m = 0 \).

- \( \tilde{C} : \mathcal{X} \to \mathbb{R}^p \), is a linear map such that \( \tilde{C}x = C_qx \) if \( x \in X_q \) and \( \tilde{C}e_{q,j} = 0 \) for each \( q \in Q, J_2 \).

- \( \tilde{B} = \{ \tilde{B}_j \in \mathcal{X} \mid j \in J \} \) is defined by \( \tilde{B}_j = x_j \in X_q \) and \( \tilde{B}_{(j,l)} = e_{q,l} \), for each \( j \in J_1, l \in J_2 \) such that \( \mu(j) = (q_j, x_j) \).

- For each \( z \in X_1, M_z : \mathcal{X} \to \mathcal{X} \) is a linear map, such that for each \( q \in Q, \forall x \in X_q : M_zx = A_{q,z}x \) and for each \( q \in Q, \forall j \in J_2, M_z e_{q,j} = B_{q,z,j} \in X_q \).

- For each \( y \in X_2, M_y : \mathcal{X} \to \mathcal{X} \) is a linear map such that \( \forall x \in X_q : M_yx = M_{\delta(q,y), y,q}x \) and \( M_y e_{q,j} = e_{\delta(q,y), j} \), for all \( q \in Q, j \in J_2 \).

Note that \( R_{HR} \) depends on the structure of the finite Moore-automaton \( A \) too.

The idea behind the choice of \( R_{HR} \) is the following. Consider the Moore-automaton interpretation of \( HR \). The representation \( R_{HR} \) can be also viewed as a Moore-automaton. We would like \( R_{HR} \) to be a realization of the continuous, i.e. \( \mathbb{R}^p \) valued part of the input-output behaviour of \( HR \). That is, if \( HR \) is a representation of some family of hybrid formal power series \( \Omega = \{ Z_j \mid j \in J \} \), then we would like \( R_{HR} \) to be a representation of \( \{ (Z_j)_C \in \mathbb{R}^p \ll X^* \gg \mid j \in J \} \). By "stacking up" the matrices \( A_{q,z}, M_{q,y,q_2} \) and taking the "state-space" \( \bigoplus_{q \in Q} X_q \), we encoded most of the information on the discrete-state dynamics which has effect on the continuous valued part of the input-output behaviour of the hybrid representation. But we still need to keep track of the elements \( B_{q,z,j} \), and for that we need to simulate the discrete-state transitions. This is done by introducing the vectors \( e_{q,j} \) and defining the action of \( M_y \) on these vectors accordingly. Of course, if \( J_2 = \emptyset \), we have no vectors \( B_{q,z,j} \) and there is no need to include \( e_{q,j} \) into the state-space of the representation \( R_{HR} \).

Recall the definition of the set \( \mathcal{O} \)

\[
\mathcal{O} = \prod_{j \in J_2} \mathbb{R}^p \ll X^* \gg
\]
Consider a hybrid representation of the form

\[ HR = (A, Y, (X_q, \{A_{q,z}, B_{q,z,j}\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2}, q \in Q, J, \mu}) \]

and assume that \( A = (Q, X_2, O, \delta, \lambda) \). Define

\[ A_{HR} = (Q, \Gamma, O \times \hat{O}, \delta, \hat{\lambda}) \]  

where \( \hat{\lambda}(q) = (\lambda(q), (T_{q,j})_{j \in J_2}) \) if \( J_2 \neq \emptyset \) and \( \hat{\lambda}(q) = (\lambda(q), \emptyset) \) if \( J_2 = \emptyset \). The realization \((\hat{A}_{HR}, \mu_D)\) will be called the finite Moore-automaton realization associated with \( HR \).

Let \( \Omega = \{Z_j \in \mathbb{R}^p \ll X^* \gg \times F(X_2^*, O) \mid j \in J\} \) be an indexed set of formal power series. Then define the indexed set of formal power series \( \Psi_\Omega \) associated with \( \Omega \) by

\[ \Psi_\Omega = \{(Z_j)_C \in \mathbb{R}^p \ll X^* \gg \mid j \in J\} \]

Define the Hankel-matrix \( H_\Omega \) of \( \Omega \) to be the Hankel-matrix \( H_{\Psi_\Omega} \) of \( \Psi_\Omega \), i.e., \( H_\Omega = H_{\Psi_\Omega} \). Define the indexed set of discrete input-output maps associated with \( \Omega \) by

\[ D_\Omega = \{\kappa_j : X_2^* \rightarrow O \times \hat{O} \mid j \in J\} \]

where the maps \( \kappa_j \) are defined as follows

\[ \kappa_j : X_2^* \ni w \mapsto ((Z_j)_D(w), (w \circ (Z_j)_C)_{1 \leq j \leq 2}) \in O \times \hat{O} \]

The following theorem describes the relationship between rationality of \( \Omega \) and rationality of \( \Psi_\Omega \) and realisability of \( D_\Omega \) by a finite Moore-automaton.

**Theorem 7.** The hybrid representation \( HR \) is a hybrid representation of the indexed set of hybrid formal power series \( \Omega \) if and only if \( R_{HR} \) is a representation of the indexed set of formal power series \( \Psi_\Omega \) and \((\hat{A}_{HR}, \mu_D)\) is a finite Moore-automaton realization of \( D_\Omega \).

Consider the following set of discrete input-output maps.

\[ \Omega_D = \{(Z_j)_D : X_2^* \rightarrow O \mid j \in J\} \]

It is easy to see that if \((\hat{A}_{HR}, \mu_D)\) is a realization of \( D_\Omega \), then \((A, \mu_D)\) is a realization of \( \Omega_D \). It is also easy to see that if \( J_2 = \emptyset \) then \((\hat{A}_{HR}, \mu_D)\) is a realization of \( D_\Omega \) whenever \((A, \mu_D)\) is a realization of \( \Omega_D \). Thus, we get the following corollary.

**Corollary 2.** Assume that \( J_2 = \emptyset \). Then \( HR \) is a hybrid representation of \( \Omega \) if and only if \( R_{HR} \) is a representation of \( \Psi_\Omega \) and \((A, \mu_D)\) is a realization of \( \Omega_D \).

Above we associated with each hybrid representation \( HR \) a representation and a finite Moore-automaton realization. Below we will present the converse of it. That is, we will associate a hybrid representation with any suitable representation and suitable finite Moore-automaton realization. The construction goes as follows.

Let \( R = (X, \{M_z\}_{z \in X}, \tilde{B}, \tilde{C}) \) be an observable representation of \( \Psi_\Omega \) and let \((\hat{A}, \hat{\zeta})\), \( \hat{A} = (Q, X_2, O \times \hat{O}, \delta, \hat{\lambda}) \) be a reachable Moore-automaton realization of \( D_\Omega \). Then define \( HR_{R, A, \zeta} \) as

\[ HR_{R, A, \zeta} = (A, Y, (X_q, \{A_{q,z}, B_{q,z}, q\}_{j \in J_2, z \in X_1}, C_q, \{M_{\delta(q,y),y,q}\}_{y \in X_2}, q \in Q, J, \mu) \]

where
\[ A = (Q, X_2, O, \delta, \Pi_O \circ \lambda) \]

- For all \( q \in Q \), let \( X_q = \text{Span}\{z \mid z \in W_q\} \) where the set \( W_q \) is defined as follows
  
  \[ W_q = \{ M_{x_{k+1}}M_{y_k}M_{x_k} \cdots M_{y_1}M_{y_i}M_{y_i-1} \cdots M_{y_2}M_{y_1}\tilde{B}(j_1,j_2) \mid \]
  
  \[ y_1, \ldots, y_k \in X_2, j_1 \in J_1, j_2 \in J_2, k \geq 0, \]
  
  \[ q = \delta(\xi(j_1), y_1 \cdots y_k), 1 \leq l \leq k + 1, x_{k+1}, \ldots, x_l \in X_1^*, \]
  
  \[ x_1 = s_1s_2, s_1 \in X_1^*, s_1 \in X_1 \} \cup \]
  
  \[ \{ M_{x_{k+1}}M_{y_k}M_{x_k} \cdots M_{y_1}M_{y_i}\tilde{B}_j \mid y_1, \ldots, y_k \in X_2, \]
  
  \[ j \in J_1, x_{k+1}, \ldots, x_l \in X_1^*, k \geq 0, q = \delta(\xi(j), y_1 \cdots y_k) \} \]

- For each \( q \in Q, z \in X_1 \), the maps \( A_{q,z} : X_q \rightarrow X_q, z \in X_1 \) are defined by \( A_{q,z} = M_z|_{X_q} \). That is, for all \( x \in X_q, z \in X_1 \),
  
  \[ A_{q,z}x = M_zx \]

- For each \( q \in Q \), the map \( C_q : X_q \rightarrow \mathbb{R} \) is defined by \( C_q = \tilde{C}|_{X_q} \). That is, for all \( x \in X_q \),
  
  \[ C_qx = \tilde{C}x \]

- For each \( q \in Q, l \in J_2, z \in X_1 \) let \( B_{q,z,l} = M_zM_w\tilde{B}_{j,l} \in X_q \) for some \( w \in X_2 \) and \( j \in J_1 \) such that \( \delta(\xi(j), w) = q \).

- For all \( q_1, q_2 \in Q, y \in X_2 \) such that \( q_1 = \delta(q_2, y) \) define the map \( M_{q_1,y,q_2} : X_{q_2} \rightarrow X_{q_1} \) as follows. For each \( x \in X_{q_2} \),
  
  \[ M_{q_1,y,q_2}x = M_yx, x \in X_{q_2} \]

- Define the map \( \mu : J_1 \rightarrow \bigcup_{q \in Q} \{ q \} \times X_q \) as follows.
  
  \[ \mu(j) = (\xi(j), \tilde{B}_j) \text{ for all } j \in J_1 \]

Notice that \( B_{q,z,l} \) is indeed well-defined for each \( q \in Q, z \in X_1, j \in J_2 \). If for some \( g, j \in J_1, w, v \in X_2, q = \delta(\xi(j), w) = \delta(\xi(g), v) \), then \( \kappa_g(v) = \kappa_j(w) \), since \( A \) is a realization of \( D_\Omega \). But then \( \kappa_g(v) = ((Z_g)D(w), (v \circ (Z_g,l)C)l_{e_j}) = ((Z_j)D(w), (w \circ (Z_j,l)C)l_{e_j}) = \kappa_j(w) \), i.e., \( v \circ (Z_g,l)C = w \circ (Z_j,l)C \). Since \( R \) is a representation of \( \Psi_\Omega \), we get that \( v \circ (Z_g,l)C(zs) = (Z_g,l)C(\tilde{C}M_sM_{l_z}M_{l_{\tilde{B}_j}}) \) for each \( s \in X_1, l \in J_2 \). Then observability of \( R \) implies that \( M_zM_w\tilde{B}_{j,l} = M_zM_w\tilde{B}_{j,l} \), thus, \( B_{q,z,l} \) is indeed well-defined. It should be clear now why we needed observability of \( R \) and reachability of \( (A, \xi) \). If \( R \) was not observable, we could have several choices for the vectors \( B_{q,z,l} \). If \( (A, \xi) \) was not reachable, we would have trouble defining \( X_q \) for the unreachable discrete states \( q \in Q \).

Notice that if \( J_2 = \emptyset \), then the construction of \( HR_{R,A,\xi} \) could be carried out for a non-observable representation \( R \) too. Assume that \( J_2 = \emptyset \) and \( (A, \xi) \) is a reachable realization of \( D_\Omega \). Assume that \( A = (Q, X_2, O, \delta, \lambda) \) and define \( A \) by

\[ \tilde{A} = (Q, X_2, O \times \tilde{O}, \delta, \tilde{\lambda}) \], where \( \tilde{\lambda}(q) = (\lambda(q), \emptyset) \)
It is easy to see that $(\bar{A}, \zeta)$ is a realization of $D_\Omega$ if $J_2 = \emptyset$. It is also easy to see that $\bar{A}$ is uniquely determined by $A$ and the construction of $HR_{R,A,\zeta}$ can be carried out based purely on the information present in $R$ and $(A, \zeta)$. Then it is justified to denote $HR_{R,A,\zeta}$ simply by $HR_{R,A}$.

The construction of $HR_{R,A,\zeta}$ in fact gives us a way to go from representations of $\Psi_\Omega$ and realizations of $D_\Omega$ to hybrid representations of $\Omega$.

**Theorem 8.** Assume that $R$ is an observable representation of $\Psi_\Omega$ and $(A, \zeta)$ is a reachable realization of $D_\Omega$. Then $HR_{R,A,\zeta}$ is a reachable hybrid representation of $\Omega$.

The remark before Theorem 8 on the construction of $HR_{R,A,\zeta}$ in the case when $J_2 = \emptyset$ yields the following corollary.

**Corollary 3.** If $J_2 = \emptyset$, $R$ is a representation of $\Psi_\Omega$ and $(A, \zeta)$ is a reachable realization of $\Omega_D$ then the hybrid representation $HR_{R,A,\zeta}$ is a reachable hybrid representation of $\Omega$.

Existence of a finite Moore-automaton realization for $D_\Omega$ is not easy to check. But we can give the following characterisation of existence of a finite Moore-automaton which is a realization of $D_\Omega$. Define the sets $W_{O,\Omega} = \{v \circ (Z_j) | v \in X_2^J, (j_1, j_2) \in J_1 \times J_2\}$ and $H_{O,\Omega} = \{(H_{O,\Omega})_{v, (j_1, j_2)} | v \in X_2^J, (j_1, j_2) \in J_1 \times J_2\}$. It is easy to see that $H_{O,\Omega}$ is simply the set of all columns of $H_\Omega$ indexed by $(v, (j_1, j_2))$ for each $v \in X_2^J$ and $(j_1, j_2) \in J_1 \times J_2$. It is also clear that there is a bijection $(H_{O,\Omega})_{v, (j_1, j_2)} \mapsto v \circ (Z_j)_{j \in J}$ from $H_{O,\Omega}$ to $W_{O,\Omega}$.

With the notation above using Theorem 1 we get the following.

**Lemma 1.** The indexed set $D_\Omega$ has a finite Moore-automaton realization if and only if $\text{card}(W_{O,\Omega}) = \text{card}(H_{O,\Omega}) < +\infty$ and $\Omega_D$ has a finite Moore-automaton realization, that is, $\text{card}(W_{\Omega_D}) < +\infty$.

That is, the lemma above states that existence of a Moore-automaton realization of $D_\Omega$ is equivalent to existence of a Moore-automaton realization of $\Omega_D$ and to $\text{card}(H_{O,\Omega}) < +\infty$, i.e. that the number of different columns of the Hankel-matrix indexed by $(v, (j_1, j_2))$, $j_2 \in J_2, j_1 \in J_1, v \in X_2^J$ is finite. The latter in fact means that the indexed set $\{(H_{O,\Omega})_{v, j} \in F(X_2^J) | j \in J_1\}$ has a Moore-automaton realization.

Theorem 1, Theorem 4, Theorem 7, Theorem 8 and Lemma 1 imply the following theorem.

**Theorem 9.** Let $\Omega$ be an indexed set of hybrid formal power series. Then the following are equivalent.

(i) $\Omega$ is rational, that is, $\Omega$ has a hybrid representation

(ii) The indexed set of formal power series $\Psi_\Omega$ is rational and $D_\Omega$ has a finite Moore-automaton realization.

(iii) rank $H_{O,\Omega} < +\infty$, card($H_{O,\Omega}$) < +\infty, and card($W_{\Omega_D}$) < +\infty

**Proof.** (i) $\implies$ (ii)

If $HR$ is a representation of $\Phi$, then from Theorem 7 it follows that $R_{HR}$ is a representation of $\Psi_\Omega$ and $(A_{HR}, \mu_D)$ is a realization of $D_\Omega$. Thus, $\Psi_\Omega$ is rational and $D_\Omega$ has a realization by a Moore-automaton.
Assume that $\Psi$ is rational and $D$ has a Moore-automaton realization. Then by Theorem 2 $D$ has a minimal Moore-automaton realization $(A,\zeta)$ and this realization is reachable and observable. Similarly, by Theorem 5 if $\Psi$ has a representation then there exists a minimal representation $R$ of $\Psi$, and $R$ is reachable and observable. Thus, $HR = HR_{R,A,\zeta}$ is well defined and by Theorem 8 $HR$ is a reachable realization of $\Phi$.

By Theorem 4, $\Psi$ is rational if and only if rank $H\Psi < +\infty$. By Lemma 1 $D$ has a Moore-automaton realization if and only if $\text{card}(W_D) < +\infty$ and $\text{card}(H O) < +\infty$.

Taking into account the discussion for the case when $J_2 = \emptyset$ we get the following corollary of the theorem above.

**Corollary 4.** Assume that $J_2 = \emptyset$. Then $\Omega$ is rational if and only if $\Psi$ is rational and $\Omega$ has a finite Moore-automaton realization. That is, $\Omega$ is rational if and only if rank $H\Omega < +\infty$ and $\text{card}(W_D) < +\infty$.

### 5.3 Minimal Hybrid Representations

Our next step will be to characterise minimal hybrid representations. We will start with characterising reachability and observability of hybrid representations. Recall from Section 4 the notion of $W$-observability for formal power series representations $R$, where $W$ is a subspace of the state-space of $R$. Consider the hybrid representation

$$HR = (A,Y, (X_q, \{A_{q,z}, B_{q,z,j} \}_{j \in J_q, z \in X_1}, C_q, \{M_{\delta(q,y),y,q} \}_{y \in X_2})_{q \in Q}, J, \mu).$$

Notice that for all $q \in Q$ the linear space $X_q$ is a subspace of the state-space of $R_{HR}$. The following lemma characterises reachability and observability of $HR$.

**Lemma 2.** The hybrid representation $HR$ is reachable if and only if $R_{HR}$ is reachable and $(A,\mu_D)$ is reachable. The hybrid representation $HR$ is observable if and only if $(\hat{A}_{HR},\mu_D)$ is observable and $R_{HR}$ is $X_q$ observable for all $q \in Q$.

Notice that if $J_2 = \emptyset$ then $(\hat{A}_{HR},\mu_D)$ is observable if and only if $(A,\mu_D)$ is observable. That is, we get the following corollary.

**Corollary 5.** If $J_2 = \emptyset$ then $HR$ is observable if and only if $(A,\mu_D)$ is observable and $R_{HR}$ is $X_q$ observable for all $q \in Q$.

It is easy to see that the following result holds too.

**Lemma 3.** If $HR$ is a hybrid representation of some indexed set of hybrid formal power series $\Omega$, then there exists a hybrid representation $HR_\tau$ of $\Omega$ such that $HR_\tau$ is reachable and $\dim HR_\tau \leq \dim HR$. Equality $\dim HR_\tau = \dim HR$ holds if and only if $HR$ is reachable.

Below we will investigate certain properties of hybrid representations of the form $HR_{R,A,\zeta}$. 
Lemma 4. Let $R$ be an observable representation of $\Psi_\Omega$, let $(\bar{A}, \zeta)$ be a reachable realization of $D_\Omega$. Consider the hybrid representation $HR = HR_{R,\bar{A},\zeta}$ and the associated representation $R_{HR}$. Then there exists a representation morphism $i_R : R_{HR} \rightarrow R$ such that $i_R(x) = x$ for all $x \in X_q$, $q \in Q$.

The lemma above has the following consequence.

Lemma 5. Assume that $R$ is minimal representation of $\Psi_\Omega$ and $(\bar{A}, \zeta)$ is a minimal realization of $D_\Omega$. Then the hybrid representation $HR = HR_{R,\bar{A},\zeta}$ is reachable and observable.

As a next step we will investigate the relationship between hybrid representation morphisms and formal power series representation and Moore-automaton morphisms. The following technical lemmas characterise the relationship between the two concepts. In fact, any hybrid representation morphism induces a representation morphism and an automaton morphism.

Lemma 6. Let $HR_1, HR_2$ be two hybrid representations and assume that

$$HR_i = (A^i, \gamma_i, (X^i_q, A^i_{q,z}, B^i_{q,z,j}), j) \in J_2, z \in X_1, C^i_q, \{M^i_{q,y,q}, y \in Q, \mu^i\}$$

$i = 1, 2$. Let $T = (T_D, T_C) : HR_1 \rightarrow HR_2$ be a hybrid representation morphism. Then there exists a representation morphism $\tilde{T} : R_{HR_1} \rightarrow R_{HR_2}$ such that $T_C(x) = T(x)$ for all $x \in X^i_{1q}$, $q \in Q$ and $\tilde{T}(e_{i,q}) = e_{T_D(q),L}$ for all $q \in Q_1$ and $L \in J_2$. The map $T_D : Q_1 \rightarrow Q_2$ is in fact an automaton morphism $T_D : (\bar{A}_{HR_1}, (\mu_1)_D) \rightarrow (\bar{A}_{HR_2}, (\mu_2)_D)$.

The following lemma is in some sense the converse of the lemma above. Let $HR = (A, \gamma, (X_q, A_{q,z}, B_{q,z,j}), j) \in J_2, z \in X_1, C_q, \{M_{q,y,q}, y \in Q, J, \mu\}$ be a hybrid representation over the index set $J$ of $\Omega$. Then the following lemma holds.

Lemma 7. Assume that $HR$ is a reachable representation of $\Omega$. Assume that $R$ is an observable representation of $\Psi_\Omega$ and $(\bar{A}, \zeta)$ is a reachable realization of $D_\Omega$. Assume that $T : R_{HR} \rightarrow R$ is a representation morphism and $\phi : (\bar{A}_{HR}, \mu_D) \rightarrow (\bar{A}, \zeta)$ is an automaton morphism. Then there exists a surjective hybrid representation morphism $H(T) = (\phi, T_C) : HR \rightarrow HR_{R,\bar{A},\zeta}$ such that for all $x \in X_{1q}$, $q \in Q$, $T_C(x) = T(x)$.

The discussion above for the case when $J_2 = 0$ yields the following corollary of Lemma 5.

Corollary 6. Assume that $J_2 = 0$. Let $R$ be any (not necessarily observable) representation of $\Psi_\Omega$ and let $(\bar{A}, \zeta)$ any reachable realization $\Omega_D$. Assume that $T : R_{HR} \rightarrow R$ is a representation morphism and $\phi : (A, \mu_D) \rightarrow (\bar{A}, \zeta)$ is an automaton morphism. Then there exists a hybrid representation morphism $H(T) : HR \rightarrow HR_{R,\bar{A},\zeta}$ such that for all $x \in X_q, q \in Q$, $T_C(x) = T(x)$.

The results of Lemma 2–7 together with Theorem 2 and Theorem 5 characterising minimality of representations and automata yield the following Theorem.

Theorem 10. If $\Omega$ has a hybrid representation, then it also has a minimal hybrid representation. Let $HR$ be a hybrid representation of $\Omega$. Then the following are equivalent.
• HR is minimal
• HR is reachable and observable

• For any reachable hybrid representation HR of Ω there exists a surjective hybrid representation morphism T : HR → HR. In particular, any two minimal hybrid representations of Ω are isomorphic.

Proof. Notice that any minimal hybrid representation is reachable. Indeed, assume that HR is a minimal hybrid representation of Ω and HR is not reachable. Then by Lemma 3 there exists a representation HRᵢ of Ω such that dim HRᵢ < dim HR and HRᵢ is reachable. Since HR is minimal, this is a contradiction.

First, we will show that if Ω has a hybrid representation, then Ω has a hybrid representation satisfying (iii). From Theorem 9 it follows that Ω has a hybrid representation if and only if ΨΩ has a Moore-automaton realization. Let R be a minimal representation of ΨΩ and (A,ζ) a minimal realization of DΩ. By Theorem 5 and Theorem 2 such a minimal representation and a minimal realization always exist. Then by Lemma 5 HR = HR⁻¹,R,A,ζ is an observable and reachable representation of Ω.

We will show that (iii) holds for HR. Indeed, if HR is a reachable hybrid representation of Ω, then RHR is reachable and (AHR’, μ’D) is reachable. By Theorem 2 and Theorem 5 there exists surjective morphisms T : RHR → R and φ : (AHR’, μ’D) → (A,ζ). Then by Lemma 7 there exists a surjective hybrid representation morphism (φ,T) : HR’ → HR such that TCX = TX for all x ∈ Xq, q ∈ Q.

Below we will show that (iii) implies (i). This will imply that HR is minimal, since HR satisfies (iii). Since HR exists whenever Ω has a hybrid representation, we get that if Ω has a hybrid representation, then it has a minimal minimal hybrid representation.

(iii) ⇒ (i)
Assume that HRᵐ satisfies (iii). Assume now that H̄R is a hybrid representation of Ω. Then by Lemma 3 there exists a reachable hybrid representation HRᵢ of Ω, such that dim HRᵢ ≤ dim H̄R. Since HRᵐ satisfies (iii) we get that there exists a surjective hybrid representation morphism T : HRᵢ → HRᵐ. It implies that dim HRᵐ ≤ dim HRᵢ ≤ dim H̄R. Thus, HRᵐ is a minimal hybrid representation of Ω.

Next we show that (ii) ⇔ (iii), and (i) ⇔ (ii).

(ii) ⇒ (iii)
Consider the realization HR = HR⁻¹,R,A,ζ above. Let HR’ be any reachable realization and consider the surjective hybrid representation morphism S = (φ,TC) existence of which was proved above. Assume that

HR’ = (A,γ,(Xq,q’,(Aq’,Bq’,ζq’),j)j∈J,q∈Q,X,Xq’,Cq’,Mq’(q,y),Mq’(q,y),y∈Xq’,q’∈Q’,J,μ’)

If HR’ is observable, then (AHR’, μ’D) is observable and RHR’ is Xq’, q ∈ Q’ observable, which implies that φ is bijective and T|Xq’ is injective for all q ∈ Q’. Since TC|Xq’ = T|Xq’ and TCX ∈ Xq if and only if x ∈ Xφ⁻¹(q), we get that TC is an isomorphism. That is, S is an hybrid representation isomorphism. It is easy to see that S⁻¹ : HR → HR’ is also a hybrid representation isomorphism,
in particular, $S^{-1}$ is surjective. For any reachable $\bar{HR}$ there exists a surjective hybrid morphism $T: \bar{HR} \rightarrow HR$. But then $S^{-1} \circ T: HR \rightarrow HR'$ is a surjective hybrid representation morphism. That is, $HR'$ satisfies (iii). Thus (ii) implies (iii).

$(i) \implies (ii)$

Indeed, let $HR_m$ a minimal hybrid representation of $\Omega$. From the discussion above it follows that $HR_m$ has to be reachable. Then there exists a surjective hybrid representation morphism $T: HR_m \rightarrow HR$. But $HR$ and $HR_m$ are both minimal, thus $\dim HR = \dim HR_m$. It implies that $T$ is a hybrid representation isomorphism. Notice that $HR$ is observable. But then by $HR_m$ has to be observable too. Thus, we get (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i). \hfill $\square$

Corollary 7. Assume that $R$ is a minimal representation of $\Psi_\Omega$ and $(\tilde{A}, \zeta)$ is a minimal realization of $D_\Omega$ (if $J_2 = \emptyset$). Then $HR_{R, \tilde{A}, \zeta}$ is a minimal hybrid representation of $\Omega$.

5.4 Partial realization theory and algorithms

In this subsection the algorithmic aspects of hybrid formal power series will be discussed. That is, we will present a procedure for constructing a hybrid representation of a family of hybrid formal power series from finite data. We will also give algorithms for checking minimality, observability and reachability of hybrid representations and for construction of a minimal hybrid representation from a specified hybrid representation. Throughout the section we will assume that $J_1$ is finite, that is, we will study only finite families of hybrid formal power series.

Recall the results on partial realization by a Moore automaton from Section 3. Recall the results on partial representation of formal power series from Section 4.

Let $\Omega = \{Z_j \in \mathbb{R}^p \ll X^* \gg \times F(X^*_2, O) \mid j \in J\}$ be an indexed set of hybrid formal power series with $J = J_1 \cup (J_1 \times J_2)$. Assume that $J_1$ is a finite set.

Consider the map $\eta_N : \mathbb{R}^p \ll X^* \gg \rightarrow \mathbb{R}^p \ll X^N \gg$, which maps $T$ to the restriction of $T$ to the set of words over $X$ of length less than $N$. That is, if $T \in \mathbb{R}^p \ll X^* \gg$, then $\eta_N(T)(w) = T(w)$ for all $w \in X^*$ such that $|w| < N$.

For each $N \in \mathbb{N}$ define the set

$\tilde{O}_N = \{((\eta_N(S_j))_{j \in J_2} \mid \forall j \in J_2 : S_j \in \mathbb{R}^p \ll X^* \gg\}$

Again, if $J_2 = \emptyset$ then we take $\tilde{O}_N = \{\emptyset\} = \tilde{O}$. For each $j_1 \in J_1$ define the map

$\kappa_{j_1, N} : X^*_2 \ni w \mapsto ((Z_{j_1})_{D}, (\eta_N(w \circ Z_{j_1, j_2}C))_{j_2 \in J_2}) \in O \times \tilde{O}_N$

Define the indexed set

$D_{\Omega, N} = \{\kappa_{j, N} \mid j \in J_1\}$

Let $H_{\Omega, N, M} = H_{\Psi_\Omega, N, M}$ for each $M, N \in \mathbb{N}$. It is easy to show that if rank $H_{\Omega, N, N} = \text{rank } H_{\Omega}$, then the restriction of the map $\eta_N$ to $W_{\Psi_\Omega}$ is a linear isomorphism. The discussion above yields the following.

Lemma 8. Assume that $(\tilde{A}, \zeta)$ is a reachable realization of $D_{\Omega, N}$, where

$A = (Q, X_2, O \times \tilde{O}_N, \delta, \lambda)$
Assume that \( \text{rank } H_{\Omega,N,N} = \text{rank } H_{\Omega} \). Consider the Moore-automaton realization \((\bar{A}, \zeta)\) such that \( \bar{A} = (Q, X_2, O \times O, \delta, \lambda) \) where

\[
\lambda(q) = (o, (\eta^{-1}_N(T_j))_{j \in J_2}) \iff \lambda(q) = (o, (T_j)_{j \in J_4})
\]

if \( J_2 \neq \emptyset \), and \( \bar{\lambda}(q) = \lambda(q) \) if \( J_2 = \emptyset \). Then \((\bar{A}, \zeta)\) is a realization of \( D_{\Omega} \). Moreover, \((\bar{A}, \zeta)\) is reachable and if \((A, \zeta)\) is observable, then \((\bar{A}, \zeta)\) is observable too.

Let \( R \) an observable representation of \( \Psi_{\Omega} \) and assume that \( H_{\Omega,N,N} = \text{rank } H_{\Omega} \). Let \((A, \zeta)\) be a reachable realization of \( D_{\Omega,N} \). Then by the lemma above \((\bar{A}, \zeta)\) is a reachable realization of \( D_{\Omega} \). Consider the hybrid representation \( HR_{R,\bar{A},\zeta} \). Notice \( A \) and \( \bar{A} \) have the same state-space and state-transition maps. Thus, all the information we need for the construction of \( HR_{R,\bar{A},\zeta} \) is already contained in \( R \) and \((A, \zeta)\). In fact, if we know \( R \) and \((A, \zeta)\), then the construction of \( HR_{R,\bar{A},\zeta} \) can be carried out by a numerical computer algorithm. Thus, denoting \( HR_{R,\bar{A},\zeta} \) simply by \( HR_{R,\bar{A},\zeta} \) is justified in some sense. In the rest of the subsection we will use this abuse of notation and we will denote \( HR_{R,\bar{A},\zeta} \) by \( HR_{R,\bar{A},\zeta} \).

The following theorem is an easy consequence of Theorem 3 and Theorem 6.

**Theorem 11.** Assume that rank \( H_{\Psi_{\Omega},N,N} = \text{rank } H_{\Psi_{\Omega},N+1,N} = \text{rank } H_{\Psi_{\Omega},N,N+1} \) and \( \text{card}(W_{D_{\Omega,N,D,D}}) = \text{card}(W_{D_{\Psi_{\Omega},N,D+1,D}}) = \text{card}(W_{D_{\Psi_{\Omega},N,D,D+1}}) \). Let \( R_N \) be the \( N \)-partial representation of \( \Psi_{\Omega} \) from Theorem 6. Let \((A_D, \zeta_D)\) be the \( D \)-partial realization of \( D_{\Omega,N} \) from Theorem 3. If \( \text{card}(W_{D_{\Omega,N,D,D}}) = \text{card}(W_{D_{\Omega,N}}) \) and rank \( H_{\Omega,N,N} = \text{rank } H_{\Omega} \) then the hybrid representation

\[
HR_{N,D} = HR_{R_N,A_D,\zeta_D}, \bar{\mu}_{R_N,A_D,\zeta_D}
\]

is a minimal hybrid representation of \( \Omega \).

Notice that \( R_N \) can be constructed from the columns of the finite matrix \( H_{\Omega,N,N} \) and \((A_D, \zeta_D)\) can be constructed from the finitely many data points of the (finite) set \( W_{D_{\Omega,N,D,D}} \). Thus, \( HR_{N,D} \) can be constructed from finitely many data and this data can be directly obtained from \( \Omega \). The following lemma is an easy consequence of Theorem 3 and Theorem 6.

**Lemma 9.** If \( \Omega \) has a hybrid representation \( HR \) such that \( \dim HR \leq (q, p) \), then \( \text{rank } H_{\Omega,M,M} = \text{rank } H_{\Omega} \) and \( \text{card}(W_{D_{\Omega,M,q,q}}) = \text{card}(W_{D_{\Psi_{\Omega},M}}) \) where \( M = q \cdot \text{card}(J_2) + p \) if \( J_2 \neq \emptyset \) and \( M = p \) otherwise. In particular, if \( \dim HR = (q, p) \) and for some \( N \in \mathbb{N} \)

\[
N \geq \begin{cases} 
q \cdot \text{card}(J_2) + p & \text{if } J_2 \neq \emptyset \\
\max\{q, p\} & \text{if } J_2 = \emptyset
\end{cases}
\]

(7)

then rank \( H_{\Omega,N,N} = \text{rank } H_{\Omega} \) and \( \text{card}(W_{D_{\Omega,N,N,N}}) = \text{card}(W_{D_{\Omega,N}}) \).

**Corollary 8.** If \( \Omega \) has a hybrid representation \( HR \) such that \( \dim HR \leq (q, p) \) then for

\[
M = \begin{cases} 
q \cdot \text{card}(J_2) + p & \text{if } J_2 \neq \emptyset \\
p & \text{if } J_2 = \emptyset
\end{cases}
\]
$HR_{M,q}$ is a minimal representation of $\Omega$. If
\[
N \geq \begin{cases} 
q \cdot \text{card}(J_2) + p & \text{if } J_2 \neq \emptyset \\
\max\{q, p\} & \text{if } J_2 = \emptyset
\end{cases}
\]
then $HR_{N,N}$ is a minimal hybrid representation of $\Omega$.

In particular, if $\Omega$ is a finite collection of hybrid formal power series it is known that $\Omega$ has a realization of dimension at most $(p,q)$, then a minimal hybrid representation of $\Omega$ can be constructed from finite data.

The results above also allow us to check reachability and observability of hybrid representations algorithmically and to construct an equivalent minimal hybrid representation from a specified representation $HR$. Consider a hybrid representation.

$HR = (A, \mathcal{Y}, (X_q, \{A_{q,z},B_{q,z,j}\})_{j \in J_2}, z \in X_1, C_q, \{M_{q,y}, y \in X_2\})_{q \in Q}, J, \mu)$

where $A = (Q, X_2, O, \delta, \lambda)$. Recall the definition of $A_{HR}$ and recall the definition of the formal power series $T_{Q,j}$, $q \in Q$, $j \in J_2$. For any $N \in N$, $N > 0$ define the following Moore-automaton

$A_{HR,N} = (Q, X_2, O \times \tilde{O}_N, \delta, \tilde{\lambda})$, and $\tilde{\lambda}(q) = (\lambda(q), (\eta_N(T_{q,j}))_{j \in J_2})$

That is, $\tilde{\lambda}(q) = (\eta_N(S_j))_{j \in J_2}$ if $\lambda(q) = (\eta_N(S_j))_{j \in J_2}$. Recall that for each $q \in Q$, $j \in J_2$, $y_1, \ldots, y_k \in X_2$, $k \geq 0$, $x_1, \ldots, x_{k+1} \in X_1^*$, $k + \sum_{z=1}^{k+1} x_z < N$

$(T_{q,j})_N(x_1 y_1 \cdots x_k y_k x_{k+1}) = C_{q,k}A_{q,\bar{z},k+1}M_{q,\bar{z},q,\bar{z},q-1} \cdots M_{q,\bar{z},q-1,q-1,\bar{z},q-1,s}B_{q,\bar{z},q-1,s}

$\tilde{\lambda}(q) = (\lambda(q), (\eta_N(T_{q,j}))_{j \in J_2})$

$\tilde{\lambda}(q) = (\lambda(q), (\eta_N(T_{q,j}))_{j \in J_2})$

with $l = \min\{|z| \mid x_z > 0\}$, $s_i \in X^*_1$, $z_i \in X_1$, $x_i = z_i s_i$ and $q_i = \delta(q, \gamma_1 \cdots \gamma_i)$, $i = 0, \ldots, k$.

**Lemma 10.** Assume the notation above. If $HR$ is a representation of $\Omega$, then $(A_{HR,N}, \mu_D)$ is a realization of $\mathcal{D}_{\Omega,N}$. The Moore-automaton $(A_{HR,N}, \mu_D)$ is reachable if and only if $(A, \mu_D)$ is reachable. Assume that $\dim HR = (p,q)$ and $N \geq q \cdot \text{card}(J_2) + p$, or, rank $H_{Q,N,N} = \text{rank } H_{Q}$ and $A$ is reachable. Then $(A_{HR,N}, \mu_D)$ is observable if and only if $(A_{HR,N})$ is observable.

Consider the following algorithm for computing $(A_{HR,N}, \mu_D)$.

**ComputeMooreAutomata**($HR, N$)

1. For each $q \in Q$, define $\tilde{\lambda}(q) = (\lambda(q), (T_{q,j})_N)_{j \in J_2}$, $(T_{q,j})_N \in \mathbb{R}^p \ll \mathbb{R}^{<N}$, $j \in J_2$.

$(T_{q,j})_N(z_1 \gamma_1 \cdots \gamma_k z_{k+1}) = C_{q,k}A_{q,\bar{z},k+1}M_{q,\bar{z},\gamma_1,k,\bar{z},q-1} \cdots M_{q,\bar{z},q-1,q-1,\gamma_1,A_{q-1,s}v,B_{q-1,s}}$

$\tilde{\lambda}(q) = (\lambda(q), (\eta_N(T_{q,j}))_{j \in J_2})$

$\tilde{\lambda}(q) = (\lambda(q), (\eta_N(T_{q,j}))_{j \in J_2})$

2. return $(Q, X_2, O \times \tilde{O}_N, \delta, \tilde{\lambda}, \mu_D)$
Since
\[(T_{q,j})^N(z_1\gamma_1 \cdots \gamma_kz_{k+1}) = C_{q_1}A_{q,kz_{k+1}}M_{q_1,\gamma_1q_2} \cdots M_{q_{k-1},\gamma_kq_{k+1}}A_{q_{k+1},s_{q_{k+1}}}B_{q_{k+1},s,j} =
\]
for all \( w = z_1\gamma_1 \cdots \gamma_kz_{k+1} \in X^* \), \( k \geq 0 \), \( z_1, \ldots, z_{k+1} \in X_1^\ast \), \( \gamma_1, \ldots, \gamma_k \in X_2 \),
\[ z_1 = \cdots = z_{l-1} = e, \ v, s \in X_1, \ |w| = k + \sum_{j=1}^{k+1} |z_j| < N, \]
it follows that ComputeMooreAutomata\((HR,N)\) always terminates and returns \((A_{HR,N}, \mu_D)\).

The following algorithm constructs \( R_{HR} \) from \( HR \). Assume that
\[ HR = (A, \mathcal{Y}, \{ A_{q,z}, B_{q,z,j} \}_{j \in J_z, z \in X_1}, C_q, \{ M_{s(q,y),y,q} \}_{y \in X_2} \}_{q \in Q} \). \]
Assume that \( Q = \{ q_1, \ldots, q_d \} \), \( card(J_2) = m \), \( J_2 = \{ j_1, \ldots, j_m \} \), \( X_q = \mathbb{R}^{n_q} \), \( q \in Q \) and \( n = n_{q_1} + n_{q_2} + \cdots + n_{q_d} \). Denote by \( \gamma_{k,l} \in \mathbb{R}^{k \times l} \) the matrix, all entries of which are zero. We will represent the state-space of \( R_{HR} \) by \( \mathbb{R}^{n+dm} \cong \mathbb{R}^n \bigoplus \mathbb{R}^{dm} \). The first \( n_q \) coordinates correspond to the space \( X_{q_1} \), the second \( n_{q_2} \) coordinates correspond to the space \( X_{q_2} \) and so on. Thus, the coordinates from \( n - n_q \) to \( n_{q_d} \) correspond to the space \( X_{q_d} \). The first \( m \) coordinates after the first \( n \) coordinates correspond the the space spanned by vectors \( \{ e_{q_1,j_1}, \ldots, e_{q_1,j_m} \} \) taken in this order. That is, the first coordinate inside the block of \( m \) coordinates correspond to \( e_{q_1,j_1} \), the second coordinate to \( e_{q_1,j_2} \) and so on. The subsequent block of \( m \) coordinates corresponds to the space spanned by \( \{ e_{q_2,j_1}, \ldots, e_{q_2,j_m} \} \), where the first coordinate inside the block corresponds to \( e_{q_2,j_1} \), the second coordinate to \( e_{q_2,j_2} \) and so on. That is, the \( i \)th coordinate in the \( i \)th block of \( m \)-coordinates corresponds to the vector \( e_{q_i,j_i} \) for all \( i = 1, \ldots, d \), \( l = 1, \ldots, m \). Here we used the notation of the definition of \( R_{HR} \) in Subsection 5.2.

ComputeRepresentation\((HR)\)

1. For all \( z \in X_1 \), define
\[
M_{e,1,z} = \begin{bmatrix}
A_{q_1,z} & 0 & 0 & \cdots & 0 \\
0 & A_{q_2,z} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{q_{k+1},z}
\end{bmatrix}
\]
\[
M_{e,2,z} = \begin{bmatrix}
\tilde{B}_{q_1,z} & 0 & 0 & \cdots & 0 \\
0 & \tilde{B}_{q_2,z} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \cdots & \tilde{B}_{q_d}
\end{bmatrix}
\]
where \( \tilde{B}_{q,z} = [B_{q,z,j_1} \ B_{q,z,j_2} \ \cdots \ B_{q,z,j_m}] \in \mathbb{R}^{n_q \times m} \) for all \( q \in Q \), \( z \in X_1 \). Let \( M_z = [M_{e,1,z} \ M_{e,2,z}] \) for all \( z \in X_1 \).

2. For all \( \gamma \in \Gamma \), define
\[
M_{\gamma,1} = \begin{bmatrix}
M_{q_1\gamma q_1} & M_{q_1\gamma q_2} & \cdots & M_{q_1\gamma q_d} \\
M_{q_2\gamma q_1} & M_{q_2\gamma q_2} & \cdots & M_{q_2\gamma q_d} \\
\vdots & \vdots & \ddots & \vdots \\
M_{q_d\gamma q_1} & M_{q_d\gamma q_2} & \cdots & M_{q_d\gamma q_d}
\end{bmatrix}
\]
\[
M_{\gamma,2} = \begin{bmatrix}
\delta_{q_1, \gamma, q_1} & \delta_{q_1, \gamma, q_2} & \cdots & \delta_{q_1, \gamma, q_4} \\
\delta_{q_2, \gamma, q_1} & \delta_{q_2, \gamma, q_2} & \cdots & \delta_{q_2, \gamma, q_4} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{q_4, \gamma, q_1} & \delta_{q_4, \gamma, q_2} & \cdots & \delta_{q_4, \gamma, q_4}
\end{bmatrix}
\]

where \( M_{q_1, \gamma, q_2} = 0 \) if \( \delta(q_2, \gamma) \neq q_1 \) and
\[
\delta_{q_1, \gamma, q_2} = \begin{cases} 
(1,1, \ldots, 1) \in \mathbb{R}^{1 \times m} & \text{if } \delta(q_2, \gamma) = q_1 \\
(0,0, \ldots, 0) \in \mathbb{R}^{1 \times m} & \text{otherwise}
\end{cases}
\]

Let \( M_\gamma = \begin{bmatrix} M_{\gamma,1} & \mathbb{Q}_{n, dm} \\ \mathbb{Q}_{n, dm} & M_{\gamma,2} \end{bmatrix} \) for all \( \gamma \in X_2 \).

3. Define
\[
\widetilde{C} = [C_{q_1}, C_{q_2}, \ldots, C_{q_d}, 0, 0, \ldots, 0]
\]

4. For all \( f \in J_1, j_l \in J_2, l = 1, \ldots, m, \) define \( \tilde{B}_{f, j_l} = e_k \), where \( k = n + (i-1)m + l, \mu_D(f) = q_i \) and \( e_k \in \mathbb{R}^{n+md} \).

5. For all \( f \in J_1 \), define \( \tilde{B}_f = \begin{bmatrix} \mathbb{Q}_{k,1} \\ \mu_{C}(f) \\ \mathbb{Q}_{n-k-n_{q_1},1} \\ \mathbb{Q}_{dm,1} \end{bmatrix} \), where \( \mu_D(f) = q_i \) and \( k = \sum_{j=1}^{i-1} n_{q_j} \).

6. return \( R = (\mathbb{R}^{n+dm}, \{X_z\}_{z \in X}, \tilde{B}, \widetilde{C}) \).

It is easy to see that the algorithm ComputeRepresentation returns a representation isomorphic to \( R_{HR_C} \).

Let \( R = (\mathbb{R}^n, \{X_z\}_{z \in X}, \tilde{B}, \widetilde{C}) \) be an observable representation of \( \Psi_\Omega \) and assume that \((\mathcal{A}, \zeta)\) is a reachable realization of \( D_{\Omega,N} \). The following algorithm constructs the hybrid representation \( H_{R,A, \zeta} \), where \( \mathcal{A} \) is constructed from \( A \) as described in Lemma 8.

\begin{enumerate}
\item Assume \( \tilde{\mathcal{A}} = (Q, X_2, O \times \tilde{O}_N, \delta, \tilde{\lambda}) \).
\item Let \( \mathcal{A} = (Q, X_2, O, \delta, \lambda), \lambda(q) = \Pi_O(\lambda(q)), \) for all \( q \in Q \).
\item Assume that \( Q = \{q_1, \ldots, q_d\} \).
\quad Let \( [U_{q_1}, U_{q_2}, \ldots, U_{q_d}] = \text{ComputeStateSpace}(R, \tilde{\mathcal{A}}, \zeta) \)
\quad where \( U_{q} \in \mathbb{R}^{n \times n_{q}} \).
\item For each \( q \in Q \), let \( \tilde{X}_q = \mathbb{R}^{n_{q}}, \) and \( \tilde{A}_{q,z} = U_{q}^T M_z U_{q}, \) for all \( z \in X_1 \).
\item For each \( q_1, q_2 \in Q, \gamma \in X_2, \delta(q_2, \gamma) = q_1 \) let \( M_{q_1, \gamma, q_2} = U_{q_1}^T M_\gamma U_{q_2} \)
\item Let \( \tilde{C}_q = CU_{q}, \) for all \( q \in Q \).
\item For each \( q \in Q \), let \( (w_q, f) = \text{ComputePath}(\tilde{\mathcal{A}}, \zeta, q) \)
\quad For all \( j \in J_2, z \in X_1 \) let \( \tilde{B}_{q,z,j} = U_{q}^T M_z M_{w_q} B_{f,j} \)
\end{enumerate}
8. For each $f \in J_1$ let $\tilde{\mu}(f) = (\zeta(f), B_f)$.

9. Let $HR = (A, \{\tilde{X}_q, \{\tilde{A}_{q,z}, \tilde{B}_{q,z,j}\}_{j \in J_2, z \in X_1}, \tilde{C}_q, \{\tilde{M}_{\delta(q,\gamma),\gamma,q}\}_{\gamma \in X_1})_{q \in Q, J, \tilde{\mu}}$.

10. return $HR$

We used the following algorithms

**ComputePath**($A, \zeta, q$)

1. $S_0 = \{ (\epsilon, q) \}$

2. $S_{k+1} = \{ (q, \gamma w) \in (Q \times X_2^+), (\delta(q,\gamma), w) \in S_k \}$

3. if there exists $(q, w) \in S_k$ such that $q = \zeta(f)$, then return $(w, f)$ else goto 2

**Proposition 6.** If $(A, \zeta)$ is reachable, then the algorithm **ComputePath**($A, \zeta, q$) terminates and it returns a pair $(w, f)$ such that $\delta(\zeta(f), w) = q$.

**ComputeStateSpace**($R, A, \zeta$)

1. Assume that $A = (Q, X_2, O, \delta, \lambda)$, and $Q = \{q_1, \ldots, q_d\}$. Assume that $R = (\mathbb{R}^ n, \{M_z\}_{z \in X_1}, B, C)$. Assume $X_1 = \{z_1, \ldots, z_p\}$.

For $i = 1, \ldots, d,$

$$(w_1, f_1) = \text{ComputePath}(A, \zeta, q_i)$$

$F_{q_i} = \{ f \in J_1 \mid \zeta(f) = q_i \}$

Assume $F_{q_i} = \{f_{i1}, \ldots, f_{ih_i}\}$ Let

$$BF_{q_i} = \begin{cases} \begin{bmatrix} B_{f_{i1}}, & B_{f_{i2}}, & \ldots, & B_{f_{ih_i}} \end{bmatrix} & \text{if } F_{q_i} \neq \emptyset \\ 0 & \text{if } F_{q_i} = \emptyset \end{cases}$$

$$R_{q_i, 0} = \begin{bmatrix} BF_{q_i}^T \\ (M_{z_1}, M_{w_{i1}} B_{f_{i1}, j_{1}})^T \\ (M_{z_2}, M_{w_{i1}} B_{f_{i1}, j_{2}})^T \\ \vdots \\ (M_{z_p}, M_{w_{i1}} B_{f_{i1}, j_{m}})^T \\ \vdots \\ (M_{z_1}, M_{w_{i2}} B_{f_{i2}, j_{1}})^T \\ (M_{z_2}, M_{w_{i2}} B_{f_{i2}, j_{2}})^T \\ \vdots \\ (M_{z_p}, M_{w_{i2}} B_{f_{i2}, j_{m}})^T \\ \vdots \\ (M_{z_1}, M_{w_{ik_i}} B_{f_{ik_i}, j_{1}})^T \\ (M_{z_2}, M_{w_{ik_i}} B_{f_{ik_i}, j_{2}})^T \\ \vdots \\ (M_{z_p}, M_{w_{ik_i}} B_{f_{ik_i}, j_{m}})^T \end{bmatrix}^{T}$$

2. For each $i = 1, 2, \ldots, d$ compute the set $\{q_{i1}, \ldots, q_{ik_i}\}$ and $\{\gamma_{i1, q_i}, \ldots, \gamma_{ik_i, q_i}\}$ such that for each $q \in Q, \delta(q, \gamma) = q_i$ if and only if $q = q_{i j}, \gamma = \gamma_{i j, q_i}$ for some $j = 1, \ldots, k_i$.

3. For each $i = 1, \ldots, d$

$$A = [R_{q_{i1}, k}, M_{z_2} R_{q_{i2}, k}, \ldots, M_{z_p} R_{q_{ik_i}, k}]$$

$$B = [M_{\gamma_{i1, q_i}} R_{q_{i1, k}}, M_{\gamma_{i2, q_i}} R_{q_{i2, k}}, \ldots, M_{\gamma_{ik_i, q_i}} R_{q_{ik_i, k}}]$$

$$R_{q_{i1}, k+1} = [A \ \ B]$$
4. If for all \( i = 1, \ldots, d \), rank \( R_{i,k+1} = \text{rank} \, R_{i,k} \) then

   (a) Compute \( U_{q_i} \in \mathbb{R}^{n \times n_{q_i}} \) such that \( n_{q_i} = \text{rank} \, R_{i,k} \), \( U_{q_i}^T U_{q_i} = Id \in \mathbb{R}^{n_{q_i} \times n_{q_i}} \) and \( \text{Im} U_{q_i} = R_{i,k} \).

   (b) return \([U_{q_1} \quad U_{q_2} \quad \ldots \quad U_{q_d}]\)

else repeat step 3

Recall the definition of a hybrid representation

\( HR_{\bar{\mathcal{A}}, \zeta} = (\mathcal{A}, \mathcal{Y}, (X_q, \{A_q, \ldots, B_q\}, j) \in J_2, z \in X, \mathcal{C}_q, \{M_s(q, y, q)\}_{y \in X} \) \)

associated with the representation \( R \) and automata \((\mathcal{A}, \zeta)\) from Section 5. With the notation above the following holds.

**Proposition 7.** The algorithm \( \text{ComputeStateSpace}(R, \bar{\mathcal{A}}, \zeta) \) always terminates and it returns the matrix \([U_{q_1} \quad \ldots \quad U_{q_d}]\) such that \( \text{Im} U_{q_i} = X_q \) and \( U_{q_i}^T U_{q_i} = I \).

Now we are ready to show that \( \text{ComputeHybridRepresentation}(R, \bar{\mathcal{A}}, \zeta) \) works correctly.

**Proposition 8.** Assume that \( R \) is an observable representation of \( \Psi_\Omega \) and \((\bar{\mathcal{A}}, \zeta)\) is a reachable realization of \( \mathcal{D}_{\Omega,N} \). Then \( \text{ComputeHybridRepresentation}(R, \bar{\mathcal{A}}, \zeta) \) always terminates. If \( R \) is a representation of \( \Psi_\Omega \) and rank \( H_{\Omega,N,N} = \text{rank} \, H_{\Omega} \), then \( \text{ComputeHybridRepresentation}(R, \bar{\mathcal{A}}, \zeta) \) returns a hybrid representation isomorphic to the hybrid representation \( HR_{R, \bar{\mathcal{A}}, \zeta} \), where \( \bar{\mathcal{A}} \) is obtained from \( \mathcal{A} \) as described in Lemma 8. That is, if \( \bar{\mathcal{A}} = (Q, X_2, O \times O_N, \delta, \bar{\lambda}) \) then \((\bar{\mathcal{A}}, \zeta)\) is a realization of \( \mathcal{D}_{\Omega} \), where \( \bar{\mathcal{A}} = (Q, X_2, O \times O, \delta, \bar{\lambda}) \) and

\[
\bar{\lambda}(q) = (O, \eta^{-1}_N(T_j))_{j \in J_2} \iff \bar{\lambda}(q) = (O, (T_j))_{j \in J_2}
\]

Notice that by Lemma 8 \((\bar{\mathcal{A}}, \zeta)\) is a reachable realization of \( \mathcal{D}_{\Omega} \).

The algorithms above enable us to formulate algorithms for minimisation, observability and reachability reduction of hybrid representations. We will also be able to present an algorithm for constructing a hybrid representation from finite data.

Consider the following algorithm

\[
\text{ComputePartialHybRepr}(H_{\Omega,N+1,N}, W_{\mathcal{D}_{\Omega,N,D}})
\]

1. Compute the \( N \) partial representation \( R \) of \( \Psi_\Omega \) from \( H_{\Omega,N+1,N} \)
2. Compute the partial Moore-automaton \( D \) realization \((\bar{\mathcal{A}}, \zeta)\) of \( \mathcal{D}_{\Omega,N} \)
3. \( HR = \text{ComputeHybridRepresentation}(R, \bar{\mathcal{A}}, \zeta) \)
4. return \( HR \)

**Proposition 9.** Assume that rank \( H_{\Omega,N,N} = \text{rank} \, H_{\Omega,N+1,N} = \text{rank} \, H_{\Omega,N,N+1} \) and \( \text{card}(W_{\mathcal{D}_{\Omega,N,D}}) = \text{card}(W_{\mathcal{D}_{\Omega,N,D+1}}) = \text{card}(W_{\mathcal{D}_{\Omega,N,D+1}}) \). The algorithm

\[
\text{ComputePartialHybRepr}(H_{\Omega,N+1,N}, W_{\mathcal{D}_{\Omega,N,D}})
\]
always terminates. If \( \text{rank } H_{\Omega,N,N} = \text{rank } H_{\Omega} \) and \( \text{card}(W_{D_{\Omega,N},D,D}) = \text{card}(W_{D_{\Omega,N}}) \) then

\[
\text{ComputePartialHybRepr}(H_{\Omega,N+1,N},W_{D_{\Omega,N},D,D})
\]

returns a minimal hybrid representation of \( \Omega \) which is isomorphic to \( HR_{N,D} \) from Theorem 11.

Using the algorithms above we can construct algorithms for minimality reduction of hybrid representations. It will also enable us to check reachability, observability of hybrid representations.

Assume that \( HR \) is a hybrid representation of \( \Omega \). The following algorithm constructs a minimal hybrid representation of \( \Omega \).

\[
\text{ComputeMinimalHybRepresentation}(HR)
\]

1. \( R = \text{ComputeRepresentation}(HR) \)
2. Assume that \( \dim HR = (q,p) \). Let \( N = qn + p \).
\[
(\tilde{A},\zeta) = \text{ComputeMooreAutomaton}(HR,N)
\]
3. Transform \( R \) to a minimal representation \( R_{\text{min}} \).
4. Transform \( (\tilde{A},\zeta) \) to a minimal Moore-automaton realization \( (A_{\text{min}},\zeta_{\text{min}}) \).
5. \( HR_{\text{min}} = \text{ComputeHybridRealization}(R_{\text{min}},A_{\text{min}},\zeta_{\text{min}}) \)
6. return \( HR_{\text{min}} \)

**Proposition 10.** The algorithm \( \text{ComputeMinimalHybRepresentation}(HR) \) above computes a minimal realization of \( \Omega \).

Reachability of \( HR \) can be checked by the following algorithm

\[
\text{IsHybReprReachable}(HR)
\]

1. \( R = \text{ComputeRepresentation}(HR) \)
2. \( (A,\zeta) = (A_{HR},\mu_D) \)
3. if \( R \) is reachable and \( (A,\zeta) \) is reachable, then return true otherwise false

It follows easily from Lemma 4 that \( \text{IsReachable}(HR) \) returns true if and only if \( HR \) is reachable and returns false otherwise. The following algorithm checks observability of \( HR \).

\[
\text{IsHybRepr Observable}(HR)
\]

1. \( R = \text{ComputeRepresentation}(HR) \)
2. Assume \( \dim R = N \).
\[
(\tilde{A},\mu_D) = \text{ComputeMooreAutomata}(HR,N)
\]
3. Compute the observability kernel $O_R$ of $R$. Using the notation of $\text{ComputeRepresentation}$ denote by $W_q, q = q_l, l = 1, \ldots, d$ the subspace

$$W_q = \{ w \in \mathbb{R}^{n+dm} | w_j = 0, \forall j \notin \sum_{z=1}^{l-1} n_{q_z} + \sum_{z=1}^{l} n_{q_z} \}$$

where $w_j$ denotes the $j$th coordinate of $w$.

If $O_R \cap W_q = \{0\}$ for all $q \in Q$ and $(A, \zeta)$ is observable, then return true, else return false.

**Proposition 11.** The algorithm $\text{IsHybReprObservable}(HR)$ always returns true if $HR$ is observable and false otherwise.

If $J_2 = \emptyset$ or $J_2 \neq \emptyset$ but we can decide whether $T_{q_1,j}(w) = T_{q_2,j}(w)$ for all $q_1, q_2 \in Q, w \in X^*, |w| < N, j \in J_2$, then the procedure $\text{ComputeMinimalHybRep}$ and procedure $\text{IsHybReprObservable}$ above can be implemented as a numerical computer algorithm. In particular, if the matrices $A_{q,z}, C_q, B_{q,z,j}, M_{q_1,y,q_2}$ are rational for all $z \in X_1, y \in X_2, q, q_1, q_2 \in Q, j \in J_2$, or $J_2 = \emptyset$, then the procedure above yields a computer algorithm for computing a minimal hybrid representation of family of hybrid formal power series.

In fact, the procedures presented above imply the following. Assume that $X_q = \mathbb{R}^{n_q}$, all matrices of $A_{q,z}, M_{q_1,y,q_2}, C_q, B_{q,z,j}$ are rational (have only rational elements) and for all $q \in Q, j \in J_2, z \in X_1, y \in X_2$ and $\mu(j)$ is a rational vector (has only rational entries) for all $j \in J_1$. Assume that $J_1$ is finite. Then the procedures $\text{IsHybReprObservable}, \text{IsHybReprReachable}$ and $\text{ComputeMinimalHybRepresentation}$ above are algorithms in the sense of classical Turing computability. That is, they can be implemented by a Turing machine. Thus, observability and reachability of hybrid representations is algorithmically decidable in this case. Similarly, minimal representation can be constructed by an algorithm.

In fact, there exists a preliminary implementation of the algorithms above in Python, which uses the numpy package for numerical computations.

### 6 Hybrid Systems

In this subsection we will present a formal definition of hybrid systems without guards. As the name indicates, a hybrid system without guards is a hybrid system where all the discrete events are externally triggered. More precisely, one could describe a hybrid system without guards as follows. The system consists of a finite state Moore-automaton, a finite collection of control systems and a collection of reset maps. We associate a control system with each state of the Moore-automaton. The states of the Moore-automaton are referred to as discrete states. The control systems are assumed to be determined by differential equations. Thus, in general, we consider nonlinear control systems, state-space of which, generally speaking is a manifold. We associate a reset map with each discrete state transitions. Reset maps are assumed to be maps between state-spaces of the control systems comprising the hybrid system. The control systems associated with the discrete states are assumed to be endowed with the input and output spaces but the state-spaces are allowed to vary with the discrete
states. The state evolution of such a hybrid system takes place as follows. One starts in a certain discrete state with a certain continuous initial state. The state trajectory evolves according to the differential equation of the control system associated with the current discrete mode, until a discrete event arrives. When a discrete event arrives, the evolution of the continuous state stops and the discrete state of the hybrid system changes according to the state transition rule of the Moore-automaton. The new continuous state is obtained by applying the reset map associated with the current discrete state transition to the continuous state where the evolution of the control stopped. All these transitions are assumed to take place instantaneously, in zero time. After the discrete state transition and resetting of the continuous state the state evolution proceeds according to the differential equation of the new discrete state, by applying the flow of the differential equation to the new continuous state. The continuous input is fed to the control system associated with the current discrete mode. The continuous output trajectory is obtained by concatenating the continuous output trajectories of the underlying continuous control systems. The discrete output trajectory is piecewise-constant, it is formed by the outputs associated with the discrete states of the Moore-automaton visited during the state-space evolution.

We assume that the discrete events and their arrival is subject to control. In other words, we assume that the discrete events are inputs and any specific discrete event can be triggered at any time. Thus, timed sequences of discrete events play the role of inputs, just as sequences of input symbols play the role of inputs for finite-state automata.

After having described in an informal way the concept of hybrid systems without guards we proceed with giving a formal definition.

**Definition 2.** A hybrid systems without guards (HSWG) is a tuple

\[ H = (A, U, Y, (X_q, f_q, h_q)_{q \in Q}, \{ R_{\delta(q,\gamma),q} \mid q \in Q, \gamma \in \Gamma \}) \]

where

- \( A = (Q, \Gamma, O, \delta, \lambda) \) is a finite-state Moore-automaton,
- \( X_q \) is a manifold for each \( q \in Q \),
- \( U \) is the set of continuous input values, it is assumed to be a manifold,
- \( Y \) is the set of continuous output values, \( Y \) is assumed to be a manifold,
- \( h_q : X_q \rightarrow Y \) is a smooth map
- \( f_q : X_q \times U \rightarrow TX_q \) is a smooth map, such that for each \( u \in U \) the map \( x \mapsto f_q(x, u) \) defines a vector field.

The set \( Q \) of states of \( A \) is called the set discrete modes, the input alphabet \( \Gamma \) of \( A \) is called the set of discrete events. The tuple \((X_q, f_q, h_q)\) can be viewed as the continuous control system associated with the discrete state \( q \in Q \). The map \( h_q \) is called the readout map. We will assume that \( f_q \) is globally Lipschitz, or more precisely, the coordinate functions are globally Lipschitz, so that the solution of the differential equation

\[ \frac{d}{dt}x(t) = f_q(x(t), u(t)) \]
is well-defined for all \( t \in \mathbb{R} \) and \( u \) piecewise-continuous functions, i.e., \( u \in PC(\mathbb{R}, U) \). In the rest of the section we will refer to hybrid systems without guards simply as hybrid systems.

Let \( \mathcal{H} = \bigcup_{q \in Q} \{ q \} \times X_q \). Let \( \mathcal{X} = \bigcup_{q \in Q} X_q \), \( A_H = \mathcal{A} \). As we already indicated at the beginning of the section, hybrid systems without guards admit two types of inputs. The inputs of the hybrid system \( H \) are functions from \( PC(T, U) \) and sequences from \( (\Gamma \times T)^* \).

The interpretation of a sequence \( (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^* \) is the following. The event \( \gamma_1 \) took place after the event \( \gamma_{i-1} \) and \( t_{i-1} \) is the elapsed time between the arrival of \( \gamma_{i-1} \) and the arrival of \( \gamma_i \). That is, \( t_i \) is the difference of the arrival times of \( \gamma_i \) and \( \gamma_{i-1} \). Consequently, \( t_i \geq 0 \) but we allow \( t_i = 0 \), that is, we allow \( \gamma_i \) to arrive instantly after \( \gamma_{i-1} \). If \( i = 1 \), then \( t_1 \) is simply the time when the event \( \gamma_1 \) arrived.

The state trajectory of the system \( H \) is a map

\[
\xi_H : \mathcal{H} \times PC(T, U) \times (\Gamma \times T)^* \times T \to \mathcal{H}
\]

of the following form. For each \( u \in PC(T, U) \), \( w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^* \), \( t_{k+1} \in T \), \( h_0 = (q_0, x_0) \in \mathcal{H} \) it holds that

\[
\xi_H(h_0, u, w, t_{k+1}) = (\delta(q_0, \gamma_1 \cdots \gamma_k), x_H(h_0, u, w, t_{k+1}))
\]

where the map \( x : T \ni t \mapsto x_H(h_0, u, w, t) \in \mathcal{X} \) is the solution of the differential equation

\[
\frac{d}{dt} x(t) = f_{q_i}(x(t), u(t + \sum_{j=1}^k t_j))
\]

where \( q_i = \delta(q_0, \gamma_1 \cdots \gamma_i) \), \( i = 1, \ldots, k \) and

\[
x(0) = x_H(h_0, u, w, 0) = R_{q_i, \gamma_i, \gamma_{i-1}}, x_H(x_0, u, (\gamma_1, t_1) \cdots (\gamma_{k-1}, t_{k-1}), t_k)
\]

if \( k > 0 \) and \( x(0) = x_0 \) if \( k = 0 \).

In fact, one can define a map \( x_H : \mathcal{H} \times PC(T, U) \times (\Gamma \times T)^* \times T \to \bigcup_{q \in Q} X_q \), by \( (h, u, s, t) \mapsto x_H(h, u, s, t) \). It is easy to see that \( \Pi_{\bigcup_{q \in Q} X_q} \circ \xi_H = x_H \). Define the set of reachable states from a subset \( \mathcal{H}_0 \subseteq \mathcal{H} \) in an obvious way as follows.

\[
R(H, \mathcal{H}_0) = \{ \xi_H(h, u, w, t) | h \in \mathcal{H}_0, u \in PC(T, U), w \in (\Gamma \times T)^*, t \in T \}
\]

We will say that the hybrid system \( H \) is reachable from \( \mathcal{H}_0 \) if \( R(H, \mathcal{H}_0) = \mathcal{H} \).

One could give an alternative definition of reachability. Define the set of continuous states reachable from \( \mathcal{H}_0 \) by

\[
\text{Reach}(H, \mathcal{H}_0) = \{ x_H(h_0, u, w, t) | u \in PC(T, U), w \in (\Gamma \times T)^*, t \in T, h_0 \in \mathcal{H}_0 \}
\]

Then \( H \) is reachable from \( \mathcal{H}_0 \) if \( \text{Reach}(H, \mathcal{H}_0) = \mathcal{X} \) and the automaton \( A_H \) is reachable from \( \Pi_{\mathcal{X}}(\mathcal{H}_0) \).

Define the function \( v_H : \mathcal{H} \times PC(T, U) \times (\Gamma \times T)^* \times O \to Y \) by

\[
v_H((q_0, x_0), u, (w, \tau), t) = \lambda(q_0, w), h_q(x_H((q_0, x_0), u, (w, \tau), t)))
\]

where \( q = \delta(q_0, w) \). For each \( h \in \mathcal{H} \) the input-output map of the system \( H \) induced by \( h \) is the function

\[
v_H(h, \cdot) : PC(T, U) \times (\Gamma \times T)^* \times T \ni (u, (w, \tau), t) \mapsto v_H(h, u, (w, \tau), t) \in O \times Y
\]
We will denote the map \((u,s,t) \mapsto \Pi_Y \circ \nu_H(h,u,s,t) \in Y\) by \(y_H(h,.)\) and we will denote \(y_H(h,.)\) simply by \(y_H(h,u,s,t)\). Two states \(h_1 \neq h_2 \in \mathcal{H}\) of the linear hybrid system \(H\) are indistinguishable if \(\nu_H(h_1,.) = \nu_H(h_2,.)\). \(H\) is called observable if it has no pair of indistinguishable states.

Throughout the paper we will mostly be concerned with realization of a set of input-output maps. It means that we will have to look at systems which have not one, but several initial states. We will use the following formalism to deal with the issue. Let \(H\) be a hybrid system and let \(\Phi \subseteq F(\mathbb{P}C(T,U) \times (\Gamma \times T)^* \times T, Y \times O)\) be a subset of the set of input-output maps. Let \(\mu : \Phi \rightarrow \mathcal{H}\) be any map. We will call the pair \((H,\mu)\) a realization. The map \(\mu\) just specifies a way to associate an initial state to each element of \(\Phi\). The statement that \((H,\mu)\) is a realization does not imply that \(H\) is realized \(\Phi\) from the set of initial states \(\text{Im} \mu\). The set \(\Phi \subseteq F(\mathbb{P}C(T,U) \times (\Gamma \times T)^* \times T, Y \times O)\) is said to be realized by a hybrid realization \((H,\mu)\) where \(\mu : \Phi \rightarrow \mathcal{H}\), if

\[
\forall f \in \Phi:\quad \nu_H(\mu(f),.) = f
\]

We will say that \(H\) realizes \(\Phi\) if there exists a map \(\mu : \Phi \rightarrow \mathcal{H}\) such that \((H,\mu)\) realizes \(\Phi\). With slight abuse of terminology, sometimes we will call both \(H\) and \((H,\mu)\) a realization of \(\Phi\). Thus, \(H\) realizes \(\Phi\) if and only if for each \(f \in \Phi\) there exists a state \(h \in \mathcal{H}\) such that \(\nu_H(h,.) = f\). We say that a realization \((H,\mu)\) is observable if \(H\) is observable and we say that \((H,\mu)\) is reachable if \(H\) is reachable from \(\text{Im} \mu\). We will denote by \(\mu_D\) the map \(\Phi \ni f \mapsto \Pi_Q(\mu(f)) \in Q\), where \(Q\) is the discrete-state space of \(H\). The map \(\mu\) can be thought of as a map which assigns to each input-output map \(f\) an initial state of the system \(H\). It is just an alternative way to fix a set of initial states. If we speak of a realization \((H,\mu)\) it will always imply that \(\text{dom}(\mu)\) is a subset of \(F(\mathbb{P}C(T,U) \times (\Gamma \times T)^* \times T, Y \times O)\), i.e. it is a set of input-output maps, and \(\mu : \text{dom}(\mu) \rightarrow \mathcal{H}\).

For a hybrid system \(H\) the dimension of \(H\) is defined as

\[
\dim H = (\text{card}(Q), \sum_{q \in Q} \dim X_q) \in \mathbb{N} \times \mathbb{N}
\]

The first component of \(\dim H\) is the cardinality of the discrete state-space, the second component is the sum of dimensions of the continuous state-spaces. For each \((m,n),(p,q) \in \mathbb{N} \times \mathbb{N}\) define the partial order relation \((m,n) \leq (p,q),\) if \(m \leq p\) and \(n \leq q\). A realization \(H\) of \(\Phi\) is called a minimal realization of \(\Phi\), if for any realization \(H'\) of \(\Phi\):

\[
\dim H \leq \dim H'
\]

The partial order relation on the dimensions of hybrid systems realizations induces a partial order on the set of all hybrid realizations. If the set of all realizations of \(\Phi\) is considered as a partially ordered set, then a minimal realization defines a minimal element of this set. Notice however, that our definition of a minimal realization is quite different from the usual definition of a minimal element of a partially ordered set. The definition of a minimal element of a partially ordered set does not imply that the minimal element is comparable (in relation) with other elements of the set. Our definition of a minimal realization explicitly requires that the minimal realization should have dimension which is smaller than the dimension of any other realization, thus, in particular, it has
to be comparable with all the realizations. That is, it is not necessarily true that any minimal element of the partially ordered set of realizations yields a minimal realization.

The reason for defining the dimension of a hybrid system as above is that there is a trade-off between the number of discrete states and dimensionality of each continuous state-space component. That is, one can have two realizations of the same input/output maps, such that one of the realizations has more discrete states than the other, but its continuous state components are of smaller dimension than those of the other system.

Let \( (H, \mu) \) and \( (H', \mu') \) be two realizations such that \( \text{dom}(\mu) = \text{dom}(\mu') \) and

\[
H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (X_q, f_q, h_q)_{q \in Q}, \{R_{q, \gamma} \mid q \in Q, \gamma \in \Gamma\})
\]

\[
H' = (\mathcal{A}', \mathcal{U}, \mathcal{Y}, (X'_q, f'_q, h'_q)_{q \in Q'}, \{R_{q, \gamma} \mid q \in Q', \gamma \in \Gamma\})
\]

where \( \mathcal{A} = (Q, \Gamma, O, \delta, \lambda) \) and \( \mathcal{A}' = (Q', \Gamma, O, \delta', \lambda') \). A pair \( T = (T_D, T_C) \) is called a hybrid system morphism from \( (H, \mu) \) to \( (H', \mu') \), denoted by \( T : (H, \mu) \rightarrow (H', \mu') \), if the following holds. The map \( T_D : (\mathcal{A}, \mu_D) \rightarrow (\mathcal{A}', \mu_D') \), where \( \mu_D(f) = \Pi_Q(\mu_D(f)) \), \( \mu_D'(f) = \Pi_Q'(\mu_D'(f)) \), is an automaton morphism and \( T_C : \bigcup_{q \in Q} X_q \rightarrow \bigcup_{q' \in Q'} X'_q \) is a map such that

- For each \( q \in Q \), the restriction \( T_C|_{X_q} : X_q \rightarrow X_{T_D(q)} \) is a smooth map
- For all \( q \in Q, x \in X_q, u \in \mathcal{U} \)

\[
D(T_C|_{X_q})(x)f_q(x, u) = f'_{T_D}(T_C(x), u) \quad \text{and} \quad h_q(x) = h'_{T_D(q)}(T_C(x))
\]

where \( D(T_C|_{X_q})(x) \) denotes the Jacobian of the smooth map \( T_C|_{X_q} \) at \( x \).

- For all \( q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1, x \in X_{q_2}, T_C(R_{q_1, \gamma, q_2}(x)) = R'_{T_D(q_1), \gamma, T_D(q_2)}(T_C(x)) \)
- \( T_C(\Pi_{X_q}(\mu(f))) = \Pi_{X_{T_D(q)}}(\mu'(f)) \) for each \( q = \mu_D(f), f \in \Phi \).

The hybrid morphism \( T \) is called a hybrid isomorphism if \( T_D \) is a bijective map and for each \( q \in Q \) the map \( T_C|_{X_q} \) is a diffeomorphism. Two hybrid system realizations are isomorphic if there exists a hybrid isomorphisms between them. Notice that a hybrid morphism can be defined only between hybrid system realizations \( (H, \mu) \) and \( (H', \mu') \) such that the domains of \( \mu \) and \( \mu' \) coincide. The following proposition gives an important system theoretic characterisation of hybrid morphisms.

**Proposition 12.** Let \( (H_i, \mu_i), i = 1, 2 \) be two hybrid systems and let \( T : (H_1, \mu_1) \rightarrow (H_2, \mu_2) \) be a hybrid morphism. Then the following holds.

\[
T \circ \xi_{H_1}(h, \cdot) = \xi_{H_2}(T(h), \cdot) \quad \text{and} \quad v_{H_1}(h, \cdot) = v_{H_2}(T(h), \cdot), \forall h \in H_1
\]

If \( T \) is an hybrid isomorphism, then \( (H_1, \mu_1) \) is reachable if and only if \( (H_2, \mu_2) \) is reachable and \( (H_1, \mu_1) \) is observable if and only if \( (H_2, \mu_2) \) is observable.
Two important subclasses of hybrid systems without guards are linear hybrid systems and bilinear hybrid systems.

**Definition 3.** A (time-invariant) linear hybrid system (abbreviated as LHS) is a hybrid system

\[ H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (X_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}) \]

such that

- For each \( q \in Q \) \( X_q = \mathbb{R}^{n_q} \), i.e. \( X_q \) has the structure of the linear space \( \mathbb{R}^{n_q} \) for some \( n_q > 0 \),
- \( \mathcal{U} = \mathbb{R}^m \) and \( \mathcal{Y} = \mathbb{R}^p \), i.e. the input and output spaces have the structure of the linear spaces \( \mathbb{R}^m \) and \( \mathbb{R}^p \), \( p, m \in \mathbb{N}, n, m > 0 \).
- For each \( q \in Q \) there exist linear maps \( A_q : X_q \rightarrow X_q \), \( B_q : \mathcal{U} \rightarrow X_q \), such that with the usual identification on \( \mathbb{R}^{n_q} \) of the tangent vectors with elements of \( \mathbb{R}^{n_q} \) the following holds
  \[ \forall x \in X_q, u \in \mathcal{U} = \mathbb{R}^m : f_q(x, u) = A_q x + B_q u \]
- For each \( q \in Q \) there exists a linear map \( C_q : X_q \rightarrow \mathcal{Y} \) such that
  \[ \forall x \in X_q : h_q(x) = C_q x \]
- The reset maps are linear, i.e., for each \( q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1 \) there exists a linear map \( M_{q_1, \gamma, q_2} : X_{q_2} \rightarrow X_{q_1} \) such that
  \[ \forall x \in X_{q_2} : R_{q_1, \gamma, q_2}(x) = M_{q_1, \gamma, q_2} x \]

We will use the following shorthand notation for linear hybrid systems

\[ H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (X_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1, \gamma, q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\}) \]

**Definition 4.** A bilinear hybrid system (abbreviated as BHS) is a hybrid system

\[ H = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (X_q, f_q, h_q)_{q \in Q}, \{R_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}) \]

such that

- For each \( q \in Q \) \( X_q = \mathbb{R}^{n_q} \), i.e. \( X_q \) has the structure of the linear space \( \mathbb{R}^{n_q} \) for some \( n_q > 0 \),
- \( \mathcal{U} = \mathbb{R}^m \) and \( \mathcal{Y} = \mathbb{R}^p \), i.e. the input and output spaces have the structure of the linear spaces \( \mathbb{R}^m \) and \( \mathbb{R}^p \), \( p, m \in \mathbb{N}, n, m > 0 \).
- For each \( q \in Q \) there exist linear maps \( A_q : X_q \rightarrow X_q \), \( B_{q,j} : X_q \rightarrow X_q \), \( j = 1, \ldots, m \) such that with the usual identification on \( \mathbb{R}^{n_q} \) of the tangent vectors with elements of \( \mathbb{R}^{n_q} \) the following holds
  \[ \forall x \in X_q, u = (u_1, \ldots, u_m)^T \in \mathcal{U} = \mathbb{R}^m, \quad f_q(x, u) = A_q x + \sum_{j=1}^m (B_{q,j} x) u_j \]
• For each $q \in Q$ there exists a linear map $C_q : \mathcal{X}_q \rightarrow \mathcal{Y}$ such that
  \[ \forall x \in \mathcal{X}_q : h_q(x) = C_q x \]

• The reset maps are linear, i.e., for each $q_1, q_2 \in Q$, $\gamma \in \Gamma$, $\delta(q_2, \gamma) = q_1$
  there exists a linear map $M_{q_1, \gamma, q_2} : \mathcal{X}_{q_2} \rightarrow \mathcal{X}_{q_1}$ such that
  \[ \forall x \in \mathcal{X}_{q_2} : R_{q_1, \gamma, q_2}(x) = M_{q_1, \gamma, q_2} x \]

We will use the following shorthand notation for bilinear hybrid systems
\[
H = (A, U, \mathcal{Y}, \{\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\ldots,m}, C_q\}_{q \in Q}, \{M_{q_1, \gamma, q_2} \mid q \in Q, \gamma \in \Gamma\})
\]

7 Linear Hybrid Systems

This section presents application of theory of hybrid formal power series to realization theory of linear hybrid systems. Subsection 7.1 recalls from [12] the definition and basic properties of linear hybrid systems. Subsection 7.2 recalls from [12] the structure of input-output maps of linear hybrid systems and the concept of hybrid kernel representation. Finally, Subsection 7.3 presents the application of hybrid formal power series to realization theory of linear hybrid systems. The results on realization theory of linear hybrid systems presented in Subsection 7.3 are essentially the same as the results described in [12], with the exception of results on partial realization. The results on partial realization theory are more general than the ones in [12]. What is really new is the application of the theory of hybrid formal power series which was developed in Section 5. The use of hybrid formal power series enables us to develop realization theory in a more concise and conceptual manner.

7.1 Basic properties

Recall from Section 6 the definition of linear hybrid systems. In this section we will introduce some additional notation and terminology, which will be used specifically for linear hybrid systems. Let
\[
H = (A, U, \mathcal{Y}, \{\mathcal{X}_q, A_q, \{B_{q,j}\}_{j=1,\ldots,m}, C_q\}_{q \in Q}, \{M_{q_1, \gamma, q_2} \mid q \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})
\]
be a linear hybrid systems. With abuse of notation denote by $\mathcal{X}$ the set $\mathcal{X} = \bigoplus_{q \in Q} \mathcal{X}_q$. Recall from Section 6 that $A_H$ refers to the Moore automaton $A$ of $H$.

Recall the definition of the continuous state-trajectory $x_H : \mathcal{H} \times PC(T, U) \times (\Gamma \times T)^* \times T \rightarrow \bigcup_{q \in Q} \mathcal{X}_q$. Notice that $\bigcup_{q \in Q} \mathcal{X}_q$ can be viewed as a subset of $\mathcal{X} = \bigoplus_{q \in Q} \mathcal{X}_q$. Thus, $x_H$ can be viewed as a map which takes its values in $\mathcal{X}$. In the sequel we will view $x_H$ as a map taking its values in $\mathcal{X}$. We can derive an explicit expression for the continuous state trajectory $x_H$ using the well-known expression for trajectories of linear systems

**Proposition 13.** For all $h_0 \in \mathcal{H}$, $h_0 = (q_0, x_0)$, $u \in PC(T, U)$, $w \in (\Gamma \in T)^*$,
\( w = (\gamma_1, t_1) \cdots (\gamma_k, t_k), \gamma_1, \ldots, \gamma_k \in \Gamma, k \geq 0, t_{k+1} \in T, \)

\[
x_H(h_0, u, w, t_{k+1}) = e^{A_{\gamma_k} t_{k+1}} M_{\gamma_k, \gamma_k, q_{k-1}} e^{A_{\gamma_k-1} t_k} \cdots M_{\gamma_1, \gamma_0} e^{A_{\gamma_0} t_0} x_0 + \\
+ \sum_{i=0}^{k} e^{A_{\gamma_k} t_{k+1}} M_{\gamma_k, \gamma_k, q_{k-1}} e^{A_{\gamma_k-1} t_k} \cdots \\
\cdots e^{A_{\gamma_i+1} t_{i+2}} M_{\gamma_i+1, \gamma_i, q_i} \int_0^{t_{i+1}} e^{A_{\gamma_i} (t_{i+1}-s)} B_q, u(s) ds
\]

where \( q_{k+1} = \delta(q_i, \gamma_{i+1}), u_i(s) = u(\sum_{j=1}^{i} t_j + s), 0 \leq i \leq k. \)

Let \( \mathcal{H}_0 \) be a subset of \( \mathcal{H} \). Recall the definition of the set \( \text{Reach}(H, \mathcal{H}_0) \). The linear hybrid system \( H \) is said to be semi-reachable from \( \mathcal{H}_0 \) if \( X \) is the vector space of the smallest dimension containing \( \text{Reach}(H, \mathcal{H}_0) \) and the automaton \( \mathcal{A}_H \) is reachable from \( \Pi_Q(\mathcal{H}_0) \). That is, \( H \) is semi-reachable from \( \mathcal{H}_0 \) if \( \mathcal{A}_H \) is reachable from \( \Pi_Q(\mathcal{H}_0) \) and \( X = \text{Span}\{z \mid z \in \text{Reach}(H, \mathcal{H}_0)\} \). Recall the notion of a hybrid system realization. Hybrid system realizations of the form \( (H, \mu) \) where \( H \) is a linear hybrid system will be called linear hybrid system realizations. We say that a linear hybrid system realization \( (H, \mu) \) is semi-reachable if \( H \) is semi-reachable from \( \text{Img} \mu \).

Recall the definition of hybrid morphisms. For linear hybrid systems we will use a related but slightly different notion of system morphism, which we will call linear hybrid morphisms. The goal of this new definition is to capture the linear structure of linear hybrid systems. Let \( (H, \mu) \) and \( (H', \mu') \) be two realizations such that \( \text{dom}(\mu) = \text{dom}(\mu') \), i.e. the domain of definition of \( \mu \) and \( \mu' \) coincide and

\[
H = (A, H, Y, (X_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1, q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})
\]

\[
H' = (A', H', Y, (X_q', A_q', B_q', C_q')_{q \in Q'}, \{M_{q_1, q_2} \mid q_1, q_2 \in Q', \gamma \in \Gamma, q_1 = \delta'(q_2, \gamma)\})
\]

where \( A = (Q, \Gamma, O, \delta, \lambda) \) and \( A' = (Q', \Gamma, O, \delta', \lambda') \). A pair \( T = (T_D, T_C) \) is called a linear hybrid morphism from \( (H, \mu) \) to \( (H', \mu') \), denoted by \( T : (H, \mu) \rightarrow (H', \mu') \), if the the following holds. The map \( T_D : (A, \mu_D) \rightarrow (A', \mu_D') \), where \( \mu_D(f) = \Pi_Q(\mu_D(f)), \mu_D'(f) = \Pi_{Q'}(\mu_D'(f)) \), is an automaton morphism and \( T_C : \bigoplus_{q \in Q} X_q \rightarrow \bigoplus_{q' \in Q'} X_q' \) is a linear morphism, such that

- \( \forall q \in Q : T_C(X_q) \subseteq X_{T_D(q)}' \),
- \( T_C A_q = A_{T_D(q)}', T_C B_q = B_{T_D(q)}', C_q = C_{T_D(q)}' T_C \) for each \( q \in Q \),
- \( T_C M_{q_1, q_2} = M'_{T_D(q_1), \gamma, T_D(q_2)} T_C, \forall q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2, \gamma) = q_1 \),
- \( T_C(\Pi_{X_q}(\mu(f))) = \Pi_{X_{T_D(q)}'}(\mu'(f)) \) for each \( q = \mu_D(f), f \in \text{dom}(\mu) \).

The linear hybrid morphism \( T \) is said to be injective, surjective or bijective if both \( T_D \) and \( T_C \) are respectively injective, surjective and bijective. Bijective linear hybrid morphisms are called linear hybrid isomorphisms. Two linear hybrid system realizations are isomorphic if there exists a linear hybrid isomorphism between them. Notice that linear hybrid morphisms can be defined
between realizations \((H, \mu)\) and \((H', \mu')\) only if \(\mu\) and \(\mu'\) have the same domain of definition.

Notice that the linear map \(T_C : \bigoplus_{q \in Q} X_q \to \bigoplus_{q' \in Q'} X'_q\) is uniquely determined by its restriction to \(\bigcup_{q \in Q} X_q\), which we will denote by \(M(T_C)\). It is easy to see that in fact \(M(T_C)\) takes it values in \(\bigcup_{q' \in Q'} X'_q\). The following proposition is an easy consequence of the remarks above.

**Proposition 14.** With the notation above, if \(T = (T_D, T_C)\) is a linear hybrid morphism, then \(\psi(T) = (T_D, M(T))\) is a hybrid morphism. Moreover, \(T\) is a linear hybrid isomorphism if and only if \(\psi(T)\) is a hybrid isomorphism.

The following proposition gives an important system theoretic characterisation of linear hybrid morphisms.

**Proposition 15.** Let \((H_i, \mu_i), i = 1, 2\) be two linear hybrid systems and let \(T : (H_1, \mu_1) \to (H_2, \mu_2)\) be a linear hybrid morphism. Then the following holds.

\(\psi(T) \circ \xi_{H_1}(h, .) = \psi(H_2)(\psi(T)h, .)\) and \(\nu_{H_1}(h, .) = \nu_{H_2}(\psi(T)h, .), \forall h \in H_1\)

If \(T\) is a linear hybrid isomorphism, then \((H_1, \mu_1)\) is semi-reachable if and only if \((H_2, \mu_2)\) is semi-reachable and \((H_1, \mu_1)\) is observable if and only if \((H_2, \mu_2)\) is observable.

### 7.2 Input-output maps of linear hybrid systems

This section deals with properties of input-output maps of linear hybrid systems. Let \(f \in \mathcal{F}(PC(T, U) \times (\Gamma \times T)^* \times T, Y \times O)\) be an input-output map. Define \(f_C = \Pi_2 \circ f : PC(T, U) \times (\Gamma \times T)^* \times T \to Y\) and \(f_D = \Pi_2 \circ f : PC(T, U) \times (\Gamma \times T)^* \times T \to O\). That is, \(f(u, w, t) = (f_C(u, w, t), f_D(u, w, t))\) for all \(u \in PC(T, U), w \in (\Gamma \times T)^*, t \in T\). Below we will define the notion of hybrid kernel representations, existence of which is an important necessary condition for existence of a linear hybrid realization.

**Definition 5 (hybrid kernel representation).** A set \(\Phi \subseteq \mathcal{F}(PC(T, U) \times (\Gamma \times T)^* \times T, Y \times O)\) is said to admit a hybrid kernel representation if there exist functions \(K^f_w : \mathbb{R}^{k+1} \to \mathbb{R}^p\) and \(G^f_{w,j} : \mathbb{R}^j \to \mathbb{R}^{p \times m}\) for each \(f \in \Phi, w \in \Gamma^*, |w| = k, j = 1, 2, \ldots, k + 1\), such that

1. \(\forall w \in \Gamma^*, \forall f \in \Phi, j = 1, 2, \ldots, |w| + 1: K^f_w\) is analytic and \(G^f_{w,j}\) is analytic
2. For each \(f \in \Phi\), the function \(f_D\) depends only on \(\Gamma^*, i.e.
\[
\forall u_1, u_2 \in PC(T, U), w \in \Gamma^*, t_1, t_2 \in T^{|w|}, t_1, t_2 \in T:
\]
\[
f_D(u_1(w, t_1), t_1) = f_D(u_2(w, t_2), t_2)
\]

The function \(f_D\) will be regarded as a function \(f_D : \Gamma^* \to O\).

3. For each \(f \in \Phi, w = \gamma_1 \gamma_2 \cdots \gamma_k \in \Gamma^*, t_{k+1} \in T, \gamma_1, \ldots, \gamma_k \in \Gamma, t = (t_1, \ldots, t_k) \in T^k:\n\[
f_C(u, (w, t), t_{k+1})) = K^f_w(t_1, \ldots, t_k, t_{k+1}) + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} G^f_{w,k+1-i}(t_{i+1}, t_{i+1} - t_i) ds + \sum_{i=1}^j \gamma_i(t_i)
\]

where \(\gamma_j(t) = u(t + \sum_{i=1}^j t_i)\).
Using the notation above, define for each \( f \in \Phi \) the function \( y'_0 : PC(T, U) \times (\Gamma \times T)^* \times T \to Y \) by

\[
y'_0(u, (w, t), t_{k+1}) = \sum_{i=0}^k \int_{t_i}^{t_{i+1}} G^f_{w,k+1-i}(t_i+1-s, t_{i+2}, \ldots, t_{k+1}) \sigma(u(s)) ds
\]

where \( t = (t_1, \ldots, t_k) \). It follows that \( y'_0(u, (w, \tau), t) = f_C(u, (w, \tau), t) - f_C(0, (w, \tau), t) \).

The intuition behind the definition for \( y'_0 \) is the following. If \((H, \mu)\) is a realization of \( \Phi \), then for each \( f \in \Phi \), \( y'_0 = \Pi_Y \circ \nu_H((\Pi_Q(\mu(f)), 0), 0) \). In fact, the following holds.

**Lemma 11.** Consider a linear hybrid system realization \((H, \mu)\)

\[
H = (A, \mathcal{U}, \mathcal{Y}, (X_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1, q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})
\]

Then \((H, \mu)\) is a realization of \( \Phi \) if and only if \( \Phi \) has a hybrid kernel representation of the form

\[
K^f_{w}(t_1, \ldots, t_{k+1}) = C_{q_k}e^{A_{q_k}t_{k+1}}M_{q_k, q_{k-1+1}} \cdots e^{A_0} \mu_C(f)
\]

\[
G^f_{w,k+j}(t_j, \ldots, t_{k+1}) = C_{q_k}e^{A_{q_k}t_{k+1}}M_{q_k, q_{k-1+1}} \cdots e^{A_{q_j+j-1}t_{k+j-1}}B_{q_{j-1}}
\]

(10)

\[
f_C(u, (w, \tau), t) = \lambda(\mu_D(f), w) \text{ for each } u \in PC(T, U), \tau \in T^k, t \in T
\]

for each \( w = \gamma_1 \cdots \gamma_k, \gamma_1, \ldots, \gamma_k \in \Gamma, k \geq 0, j = 1, \ldots, k + 1, f \in \Phi \). If \((H, \mu)\) is a realization of \( \Phi \), then \( y'_0 = \Pi_Y \circ \nu_H((\mu_D(f), 0), 0) \).

If the set \( \Phi \) has a hybrid kernel representation, then the collection of analytic functions \( \{K^f_{w}, G^f_{w,j} \mid w \in \Gamma^*, j = 1, 2, \ldots, \mid w \mid + 1, f \in \Phi \} \) determines \( \{f_C \mid f \in \Phi \} \). Since \( K^f_{w} \) is analytic, we get that the collection \( \{D^\alpha K^f_{w}, D^\beta G^f_{w,j} \mid \alpha \in N^{|w|}, \beta \in N^j \} \) determines \( K^f_{w} \) and \( G^f_{w,j} \) locally.

For each \( f \in \Phi, u \in PC(T, U), w \in \Gamma^* \) define the maps

\[
f_C(u, (w, .)) : T^{|w|+1} \ni (t_1, \ldots, t_{|w|+1}) \mapsto f_C(u, (w, t_1 \cdots t_{|w|}), t_{|w|+1})
\]

\[
y'_0(u, (w, .)) : T^{|w|+1} \ni (t_1, \ldots, t_{|w|+1}) \mapsto y'_0(u, (w, t_1 \cdots t_{|w|}), t_{|w|+1})
\]

By applying the formula \( \frac{d}{dt} \int_0^t f(t, \tau) d\tau = f(t, t) + \int_0^t \frac{d}{dt} f(t, \tau) d\tau \) and Definition 5 one gets

\[
D^\alpha K^f_{w} = D^\alpha f_C(0, w, .), D^\beta G^f_{w,j} e_z = D^\beta y'_0(e_z, w, .) \quad (11)
\]

where \( w = \gamma_1 \cdots \gamma_k, l \leq k + 1, N^{k+1} \ni \beta = (0, 0, \ldots, 0, \xi_1 + 1, \xi_2, \ldots, \xi_l), \)

and \( e_z \) is the \( z \)th unit vector of \( \mathbb{R}^m \), i.e. \( e_z^T e_j = \delta_{zj} \). The formula above implies that all the high-order derivatives of the functions \( K^f_{w}, G^f_{w,j} \ (f \in \Phi, w \in \Gamma^*, j = 1, 2, \ldots, |w| + 1) \) at zero can be computed from high-order derivatives of the functions from \( \Phi \) with respect to the relative arrival times of discrete events.

From the discussion above one gets the following.
**Proposition 16.** Let $\Phi \subseteq F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$. Let $(H, \mu)$ be a linear hybrid system realization

$$H = (A, \mathcal{U}, \mathcal{Y}, (X_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1, q_2} | q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$$

where $A = (Q, \Gamma, O, \delta, \lambda)$. The pair $(H, \mu)$ is a realization of $\Phi$ if and only if $\Phi$ has a hybrid kernel representation and for each $w \in \Gamma^*$, $f \in \Phi$, $j = 1, 2, \ldots, m$ and $\alpha \in \mathbb{N}^{||w||+1}$ the following holds

$$D^\alpha f_{\gamma_0}(e_j, w, \ldots) = D^\beta G_{w, k+2-l}^f \gamma_j = C_{q_k A^\alpha_{k+1} M_{q_k, \gamma_k} \cdot \cdots \cdot M_{q_l, \gamma_l} A^{\alpha_l - 1} B_{q_l-1} e_j$$

$$D^\alpha f_{\gamma_0}(0, w, \ldots) = D^\alpha K_w = C_{q_k A^\alpha_{k+1} M_{q_k, \gamma_k} \cdot \cdots \cdot M_{q_l, \gamma_l} A^{\alpha_l} x_0$$

$$f_D(w) = \lambda(q_0, w)$$

where $l = \min\{h | \alpha_h > 0\}$, $e_z$ is the $z$th unit vector of $\mathcal{U}$, $\beta = (\alpha_1 - 1, \ldots, \alpha_{||w||+1})$ and $w = \gamma_1 \cdots \gamma_k, \gamma_1, \ldots, \gamma_k \in \Gamma$, $q_j = \delta(q_0, \gamma_1 \cdots \gamma_j)$ and $\mu(f) = (q_0, x_0)$.

### 7.3 Realization of input-output maps by linear hybrid systems

In this section the solution to the realization problem will be presented. That is, given a set of input-output maps we will formulate necessary and sufficient conditions for the existence of a linear hybrid system realizing that set. In addition, characterisation of minimal systems realizing the specified set of input-output maps will be given. We will use the theory of hybrid formal power series developed in Section 5.

The main idea behind the realization construction is the following. We associate a family of hybrid formal power series with the specified set of input-output maps. It turns out that if the set of input-output maps admits a hybrid kernel representation, then there is a one-to-one correspondence between the linear hybrid systems realization of the set of input-output maps and the hybrid representations of the hybrid formal power series. Moreover, minimal linear hybrid realizations correspond to minimal hybrid representations. Thus, we can use the theory of hybrid representations developed in Section 5 to develop realization theory for linear hybrid systems.

The outline of the subsection is the following. We start with presenting necessary and sufficient conditions for observability and semi-reachability of linear hybrid systems. Then we will proceed with defining the family of hybrid formal power series associated with the set of input-output maps and the correspondence between linear hybrid realizations and hybrid representations. As it was explained before, this correspondence will be used to formulate necessary and sufficient conditions for existence of a linear hybrid realization and to characterise minimality.

#### 7.3.1 Observability and semi-reachability of linear hybrid systems

The following two theorems characterise observability and semi-reachability of linear hybrid systems. Observability of related classes of hybrid systems was investigated in [17, 3, 4]. Let

$$H = (A, \mathcal{U}, \mathcal{Y}, (X_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1, q_2} | q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})$$
be a linear hybrid system. The following theorem characterises observability of linear hybrid systems.

**Theorem 12.** $H$ is observable if and only if

1. For each $s_1, s_2 \in Q$, $s_1 = s_2$ if and only if for all $\gamma_1, \ldots, \gamma_k \in \Gamma, j_1, \ldots, j_{k+1} \geq 0, 0 \leq l \leq k, k \geq 0$:
   
   $$\lambda(s_1, \gamma_1 \cdots \gamma_k) = \lambda(s_2, \gamma_1 \cdots \gamma_k) \quad \text{and}$$
   
   $$C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_{i+1}, \gamma_{i+1}, q_i} A_{q_i}^{j_i+1} B_{q_i} =$$
   
   $$= C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_{i+1}, \gamma_{i+1}, q_i} A_{q_i}^{j_i+1} B_{q_i}$$

   where $q_j = \delta(s_1, \gamma_1 \cdots \gamma_j)$ and $v_j = \delta(s_2, \gamma_1 \cdots \gamma_j), j = 0, 1, \ldots, k$.

2. For each $q \in Q$ it holds that $O_{H,q} := \bigcap_{w \in \Gamma} O_{q,w} = \{0\} \subseteq X_q$ where $\forall w = \gamma_1 \cdots \gamma_k \in \Gamma^*, \gamma_1, \ldots, \gamma_k \in \Gamma, k \geq 0$:
   
   $$O_{q,w} = \bigcap_{j_1, \ldots, j_{k+1} \geq 0} \ker C_{q_k} A_{q_k}^{j_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_{i+1}, \gamma_{i+1}, q_i} A_{q_i}^{j_i+1}$$

   where $q \in Q, q_0 = \delta(q_0, \gamma_1 \cdots \gamma_l), 0 \leq l \leq k, k \geq 0$.

A quick look at Proposition 2 from Section 5 reveals that the conditions for observability of linear hybrid systems described in the theorem above are very similar to the conditions for observability of hybrid representations. It is by no means a coincidence and it is related to the correspondence between linear hybrid realizations and hybrid representations. More precisely, there is a direct correspondence between observability of linear hybrid systems and observability of certain hybrid representations. We will present this correspondence later on in this section. The following theorem characterises semi-reachability of $(H, \mu)$.

**Theorem 13.** $(H, \mu)$ is semi-reachable if and only if $(A_H, \mu_D), \mu_D = \Pi_Q \circ \mu$, is reachable and $\dim W_H = \sum_{q \in Q} \dim X_q$, where

$$W_H = \text{Span}\{A_{q_k}^{j_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_{i+1}, \gamma_{i+1}, q_i} A_{q_i}^{j_i+1} B_{q_i,u}, \mu_D(x) = \mu(f), f \in \Phi$$

$$j_1, \ldots, j_{k+1} \geq 0, u \in U, \gamma_1, \ldots, \gamma_k \in \Gamma, (q_f, x_f) = \mu(f), f \in \Phi, q_j = \delta(q_0, \gamma_1 \cdots \gamma_j), 0 \leq l \leq k, j \geq 0}$$

$$\subseteq \bigoplus_{q \in Q} X_q$$

Later we will show that observability and semi-reachability of linear hybrid systems can be checked algorithmically.

### 7.3.2 Realization theory of linear hybrid systems

Let $\Phi \subseteq F(\mathcal{PC}(T,U) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)$ be a set of input-output maps. Assume that $\Phi$ has a hybrid kernel representation. Then Proposition 16 allows us to reformulate the realization problem in terms of rationality of certain hybrid formal power series. The construction of these hybrid formal power series goes as follows.
Let $\tilde{\Gamma} = \Gamma \cup \{e\}, e \notin \Gamma$. Every $w \in \tilde{\Gamma}^*$ can be written as $w = e^{\alpha_1} \gamma_1 e^{\alpha_2} \gamma_2 \cdots \gamma_k e^{\alpha_{k+1}}$ for some $\gamma_1, \ldots, \gamma_k \in \Gamma$, $\alpha_1, \ldots, \alpha_{k+1} \geq 0$. For each $f \in \Phi$ define the formal power series $(Z_f)_C, (Z_{f,j})_C \in \mathbb{R}^p \ll \tilde{\Gamma}^* \gg, j = 1, \ldots, m$ as follows.

$$(Z_f)_C(e^{\alpha_1} \gamma_1 e^{\alpha_2} \gamma_2 \cdots \gamma_k e^{\alpha_{k+1}}) = D^{\alpha} f_C(0, w, \cdot)$$

$$(Z_{f,j})_C(e^{\alpha_1} \gamma_1 e^{\alpha_2} \gamma_2 \cdots \gamma_k e^{\alpha_{k+1}}) = D^{\alpha} y_0^f(e_j, w, \cdot)$$

where $w = \gamma_1 \cdots \gamma_k$ and $\alpha = (\alpha_1, \ldots, \alpha_{k+1}) \in \mathbb{N}^k$. Notice that $(Z_{f,j})_C(v) = 0$ for all $v \in \Gamma^*$. Notice that the complete knowledge of the functions $K_{f,0}$ and $G_{w,t}$ is not needed in order to construct the formal power series $(Z_f)_C, (Z_{f,j})_C$. In fact, one can think of $(Z_f)_C$ as an object containing all the information on the behaviour of $f$ with the zero continuous input. The series $(Z_{f,j})_C, j = 1, \ldots, m$, contains all the information on the behavior of the pair $(q, 0)$, where $q$ is the discrete part of the hybrid state inducing $f$ in some realization of $\Phi$ (if there is any).

Let $J = I_\Phi = \Phi \cup (\Phi \times \{1, 2, \ldots, m\})$. That is, $J$ can be interpreted as a hybrid power series index set, where $J_1 = \Phi$ and $J_2 = \{1, \ldots, m\}$. The alphabet $\tilde{\Gamma}$ decomposes into two disjoint subsets $\Gamma$ and $\{e\}$. With the notation of Section 5, let $X = \tilde{\Gamma}, X_1 = \{e\}, X_2 = \Gamma$. Define the hybrid formal power series $Z_f$ and $Z_{f,j}, j = 1, \ldots, m$ by

$$Z_f = (Z_C, f_D) \text{ and } Z_{f,j} = ((Z_{f,j})_C, f_D)$$

That is, the discrete-valued part of the hybrid formal power series $Z_f$ and $Z_{f,j}, j \in \{1, \ldots, m\}$ is the map $f_D$, i.e. the discrete-valued part of $f \in \Phi$. Notice that $\Phi$ has to have a hybrid kernel representation for $f_D$ to be a map from $\Gamma^*$ to $O$. The continuous valued parts of $Z_f$ and $Z_{f,j}$ are the formal power series $(Z_f)_C$ and $(Z_{f,j})_C$ respectively. Thus, the continuous valued parts store the high-order derivatives at zero of $f_C(0, \cdot)$ and $y_0^f(e_j, \cdot), j = 1, \ldots, m$. By analyticity of $f_C(0, \cdot)$ and $y_0^f(e_j, \cdot)$ these high-order derivatives determine the functions uniquely. Thus, by the particular structure of $f$ imposed by existence of a hybrid kernel representation we get that $(Z_f)_C$ and $(Z_{f,j})_C, j = 1, \ldots, m$ determine $f_C$ completely, thus the hybrid formal power series $Z_f$ together with $Z_{f,j}$ determine $f$ completely.

Note that we used heavily the assumption that $\Phi$ has a hybrid kernel representation while constructing the hybrid formal power series $Z_f$ and $Z_{f,j}, j = 1, \ldots, m$. In particular, if $\Phi$ does not have a hybrid kernel representation, then the derivatives of $f(0, \cdot)$ or $y_0^f(e_j, \cdot)$ need not exist or $f_D$ might depend on switching times or continuous inputs instead of sequences of discrete inputs only.

We will use the hybrid formal power series above to associate with $\Phi$ a suitable family of hybrid formal power series. Define the set of hybrid formal power series associated with $\Phi$ by

$$\Psi_\Phi = \{Z_j \in \mathbb{R}^p \ll \tilde{\Gamma}^* \gg \times F(\Gamma^*, O) \mid j \in I_\Phi\}$$

It is easy to see that $\Psi_\Phi$ is a well-posed indexed set of hybrid formal power series. Define the Hankel-matrix $H_\Phi$ of $\Phi$ as $H_\Phi = H_{\Phi, \Phi}$. Notice that if $\Phi$ is finite, then $\Psi_\Phi$ has finitely many elements.
Let \((H, \mu)\) be a hybrid system realization with \(\mu : \Phi \rightarrow \bigcup_{q \in Q} \{q\} \times X_q\). Define the hybrid representation \(HR_{H, \mu}\) associated with \((H, \mu)\) by

\[
HR_{H, \mu} = (A, X, (X_q, \{A_{q,z}, B_{q,z,j}\}_{j \in J_2, z \in X_1}, C_q, \{M_{y,q,y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)
\]

where \(J = I_\Phi, J_1 = \Phi, J_2 = \{1, \ldots, m\}\), \(X_1 = \{e\}\), \(X_2 = \Gamma\) and for each \(q \in Q, j = 1, \ldots, m\)

\[
A_{q,e} = A_q \text{ and } B_{q,e,j} = B_q e_j
\]

where \(e_j\) is the \(j\)th unit vector of \(U\).

Conversely, let \(HR = (A, Y, (X_q, \{A_{q,z}, B_{q,z,j}\}_{j \in J_2, z \in X_1}, C_q, \{M_{y,q,y,q}\}_{y \in X_2})_{q \in Q}, J, \mu)\) be a hybrid representation with index set \(I_\Phi\) such that \(X_1 = \{e\}\), \(X_2 = \Gamma\), \(J_1 = \Phi, J_2 = \{1, \ldots, m\}\). Define the linear hybrid realization \((HR_{HR}, \mu_{HR})\) associated with \(HR\) as follows

\[
HR_{HR} = (A, U, (X_q, \{A_{q,z}, B_q, C_q\}_{q \in Q}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})
\]

and

\[
\mu_{HR} = \mu
\]

where for each \(q \in Q\)

\[
A_q = A_{q,e} \text{ and } B_q = [B_{q,e,1} \quad B_{q,e,2} \cdots B_{q,e,m}]
\]

It is easy to see that \((H_{HR_{HR}}, \mu_{HR_{HR}}) = (H, \mu)\) and \(HR_{HR, \mu_{HR}} = HR\) for any hybrid representation \(HR\) and linear hybrid realization \((H, \mu)\). It is also easy to see that \(\dim H = \dim HR_{HR, \mu}\).

The following theorem follows easily from Proposition 16 and plays a crucial role in realization theory of linear hybrid system.

**Theorem 14.** A linear hybrid system \((H, \mu)\) is a realization of \(\Phi\) if and only if \(\Phi\) has a hybrid kernel representation and \(HR_{H, \mu}\) is a hybrid representation of \(\Psi_\Phi\). Conversely, if \(\Phi\) has a hybrid kernel representation and \(HR\) is a hybrid representation of \(\Psi_\Phi\) then \((HR_{HR}, \mu_{HR})\) is a linear hybrid system realization of \(\Phi\).

The theorem above allows us to reduce the realization problem for linear hybrid systems to existence of a hybrid representation of a indexed set of hybrid formal power series. Moreover, Theorem 12 and Theorem 13 allow us to relate observability and semi-reachability of linear hybrid systems to observability and reachability of hybrid representations.

**Theorem 15.** A linear hybrid system realization \((H, \mu)\) is observable if and only if \(HR_{H, \mu}\) is observable. A linear hybrid system realization \((H, \mu)\) is semi-reachable if and only if \(HR_{H, \mu}\) is reachable.

Notice that both \(H\) and \(HR_{H, \mu}\) have the same state-space. It is easy to see that the following holds.
Lemma 12. Let \((H_i, \mu_i)\), \(i = 1, 2\) be a two linear hybrid system realizations, the map \(T : (H_1, \mu_1) \rightarrow (H_2, \mu_2)\) is a linear hybrid morphism, then \(T\) is also a \(T : HR_{H_1, \mu_1} \rightarrow HR_{H_2, \mu_2}\) hybrid representation morphism. Conversely, if \(T : HR_1 \rightarrow HR_2\) is a hybrid representation morphism then \(T\) can be viewed as a \(T : (HR_1, \mu_{HR_1}) \rightarrow (HR_2, \mu_{HR_2})\) linear hybrid morphism. The map \(T\) is a surjective, injective, isomorphism as a linear hybrid morphism if and only if \(T\) is surjective, injective, isomorphism as a hybrid representation morphism.

Recall from Section 5 the definitions of \(H_O, \Omega, D\) for an indexed set of hybrid formal power series \(\Omega\). Let \(H_O, \Phi = H_O, \Psi \Phi, \Phi D = (\Psi \Phi)_D\). From the discussion above, using the results on theory of hybrid formal power series, namely Theorem 9 and Theorem 10, we can derive the following result.

Theorem 16 (Realization of input/output map). Let \(\Phi \subseteq F(\mathcal{PC}(T,U) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times O)\). The following are equivalent.

(i) \(\Phi\) has a realization by a linear hybrid system,
(ii) \(\Phi\) has a hybrid kernel representation, \(\Psi \Phi\) is rational
(iii) \(\Phi\) has a hybrid kernel representation, \(\text{rank } H_\Phi < +\infty, \text{card}(W_{\Phi D}) < +\infty, \text{and card}(H_{\Phi O}) < +\infty\).

We can also characterise minimal linear hybrid realizations.

Theorem 17 (Minimal realization). If \(\Phi\) has a linear hybrid system realization, then it has a minimal linear hybrid system realization. If \((H, \mu)\) is a realization of \(\Phi\), then the following are equivalent.

(i) \((H, \mu)\) is minimal,
(ii) \((H, \mu)\) is semi-reachable and it is observable,
(iii) For each \((H', \mu')\) semi-reachable linear hybrid system realization of \(\Phi\) there exists a surjective linear hybrid morphism \(T : (H', \mu') \rightarrow (H, \mu)\). In particular, all minimal hybrid linear systems realizing \(\Phi\) are isomorphic.

The theory of hybrid formal power series developed in Section 5 allows us to formulate a partial realization theorem for linear hybrid systems. It also enables us to formulate algorithms for deciding observability and semi-reachability of linear hybrid systems and to give an algorithm for constructing a minimal linear hybrid system realization based on a specified linear hybrid system realization.

Let \(\Phi\) be a set of input-output maps and assume that \(\Phi\) has a hybrid kernel representation. Our first objective is to construct a linear hybrid system realization of \(\Phi\) from finitely many data points. It is easy to see that all information needed for constructing the indexed set of hybrid formal power series \(\Omega = \Psi \Phi\) can be obtained (in theory) from the set of input-output maps \(\Phi\). In the remaining part of the section we will tacitly assume that \(\Phi\) is finite, i.e., \(\Phi\) consists of finitely many input-output maps.

Recall the results of Subsection 5.4. If \(\Phi\) is a finite collection of input-output maps, then the index set \(J = \Phi \cup (\Phi \times \{1, \ldots, m\})\) of \(\Psi \Phi\) is finite. It is easy to see that if \(\Phi\) is finite then all the data for constructing \(W_{\Phi \Phi, N, D, D}\) and \(H_{\Phi \Phi, N, N}\) can be obtained from the input-output maps of \(\Phi\) and the number of data points needed for constructing \(W_{\Phi \Phi, N, D, D}\) and \(H_{\Phi \Phi, N, N}\) is finite. Theorem 11 yields...
that the finite data from \( W_{\Phi_{\Psi \Phi}} \) and \( H_{\Phi_{\Psi \Phi}} \) can be used to compute a minimal hybrid representation of \( \Psi_{\Phi} \). But any minimal hybrid representation \( HR \) of \( \Psi_{\Phi} \) yields a minimal linear hybrid realization \( (H_{HR}, \mu_{HR}) \) of \( \Phi \). Thus, we get the following result. Let \( H_{\Phi,N,M} = H_{\Phi, N, M}, D_{\Phi,N} = D_{\Phi, N} \) for all \( N, M \in \mathbb{N}, N, M > 0 \).

**Theorem 18.** Assume that \( \Phi \) is a finite collection of input-output maps and \( \Phi \) has a hybrid kernel representation. Assume that \( \text{rank } H_{\Phi,N,N} = \text{rank } H_{\Phi} \) and \( \text{card}(W_{\Phi_{\Psi \Phi}}) = \text{card}(W_{\Phi_{\Psi \Phi}}) \). Let \( H_{R,N,D} \) be the hybrid representation from Theorem 11. Then \( (H_{R,N,D}, \mu_{R,N,D}) = (H_{HR,N,D}, \mu_{HR,N,D}) \) is a minimal linear hybrid system realization of \( \Phi \) and it can be constructed from finite data which can be obtained directly from \( \Phi \). In particular, if \( \Phi \) has a linear hybrid system realization \((H, \mu)\) such that \( \dim H = (p, q) \) and \( pq + p \leq N \), then \((H_{R,N,D}, \mu_{R,N,D})\) is a minimal linear hybrid system realization of \( \Phi \) and it can be constructed from finitely many data which is directly obtainable from \( \Phi \).

The results of Subsection 5.4 also allow us to check observability and semi-reachability of linear hybrid systems algorithmically. Indeed, consider a linear hybrid system realization \((H, \mu)\). It is easy to see that the construction of \( H_{R,H,\mu} \) can be carried out by a computer algorithm. It follows that \( H_{R,H,\mu} \) is reachable if and only if \((H, \mu)\) is semi-reachable and \( H_{R,H,\mu} \) is observable if and only if \( H \) is observable. The procedures \( \text{IsHybRepObservable} \) and \( \text{IsHybRepReachable} \) can be carried out algorithmically. Thus, if all the entries of the system matrices of \( H \) are rational and all the values of \( \mu \) are rational, then observability and semi-reachability of \((H, \mu)\) is algorithmically decidable and a minimal linear hybrid realization of \( \Phi \) can be constructed from \((H, \mu)\) by an algorithm in sense of classical Turing computability.

As an illustration we will present below a numerical example.

**Example**

Consider the following linear hybrid system. Consider the Moore-automaton \( A = (Q, \Gamma, O, \delta, \lambda) \), where \( Q = \{q_1, q_2\} \), \( \Gamma = \{a, b\} \) and \( O = \{0\} \). Define the discrete state transition map by \( \delta(q_1, a) = q_1, \delta(q_1, b) = q_2, \delta(q_2, b) = q_2, \delta(q_2, a) = q_2 \). Define the readout map \( \lambda(q_1) = \lambda(q_2) = 0 \). Consider the linear hybrid system

\[
H = (A, U, Y, (X_q, A_q, B_q, C_q)_{q \in Q}, \{M_{q_1, \gamma, q_2} \mid q_1, q_2 \in Q, \gamma \in \Gamma, q_1 = \delta(q_2, \gamma)\})
\]

where \( U = \mathbb{R}, p = m = 1, X_{q_1} = \mathbb{R}^1 \) and \( X_{q_2} = \mathbb{R}^2 \) and the matrices \( A_q, B_q, C_q, q \in \{q_1, q_2\} \) are of the following form

\[
A_{q_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad B_{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C_{q_1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
\]

\[
A_{q_2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad B_{q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}
\]
The linear reset maps are the following

\[ M_{q_1,a,q_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{q_2,b,q_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ M_{q_1,a,q_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_{q_2,b,q_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

The form of the input/output map \( \nu_H((q_2, x_0),.) \) induced by \((q_2, x_0), x_0 = [1 \ 0]^T \) is quite complex, as a demonstration we will present below the output to the discrete input sequence \((b, t_1)(a, t_2)(a, t_3)(b, t_4)\).

\[ v_H((q_2, x_0), u, (b, t_1)(a, t_2)(a, t_3)(b, t_4), t_5) = (a, e^{2 t_5} e^{3 t_4} e^{3 t_3} e^{2 t_2} e^{2 t_1} + \int_{0}^{t_1+\cdots+t_5} e^{t_1+\cdots+t_5-s} u(s) ds) \]

Consider a linear hybrid system \( H_m \) of the following form

\[(A^m, U, Y, (X_q^m, A_q^m, B_q^m, C_q^m)_{q \in Q^m}, \{M_{q_1,\gamma,q_2} \mid q_1, q_2 \in Q^m, q \in \Gamma, q_1 = \delta^m(q_2, \gamma)\})\]

where \( Q^m = \{q\} \), \( X_q^m = \mathbb{R}^3 \), the automaton \( A^m = (Q^m, \Gamma, O, \delta^m, \lambda^m) \) is given by

\[ \delta^m(q, z) = q, z \in \{a, b\} \text{ and } \lambda^m(q) = a \]

The matrices \( A_q^m, B_q^m, C_q^m, M_{q_1,q_2,q}^m, z \in \{a, b\} \) are specified below

\[ A_q^m = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_q^m = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad C_q^m = \begin{bmatrix} -1 & -1 & -1.414214 \end{bmatrix} \]

\[ M_{q_1,q_2,q}^m = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_{q_2,q_1,q}^m = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Define \( \mu_m((q_2, x_0),.) = (q, z_0) \) by \( z_0 = [-0 -0 -0.707107]^T \). Then \( (H_m, \mu_m) \) is a minimal linear hybrid system realization of \( v_H((q_2, x_0),.) \). The realization \( H_m \) was computed using a Matlab implementation of the algorithm presented in the paper.

## 8 Bilinear Hybrid Systems

This section presents application of hybrid formal power series theory to realization theory of bilinear hybrid systems. Subsection 8.1 recalls the definition and basic properties of bilinear hybrid systems. The material of this subsection can be found in [9]. Subsection 8.2 reviews the properties of input-output maps of bilinear hybrid systems and the notion of hybrid Fliess-series expansion. Again, the presented results are essentially the same as in [9]. Finally, Subsection 8.3 develops realization theory of bilinear hybrid systems by using the theory of hybrid formal power series developed in Section 5. Again, most of the results of Subsection 8.3 can be found in [9]. The real novelty lies in application of hybrid formal power series.
8.1 Definition and basic properties

Recall from Section 6 the definition of bilinear hybrid systems. Similarly to ordinary bilinear systems, the trajectory of a hybrid bilinear system admits a representation by an absolutely convergent series of iterated integrals.

Before giving the precise formulation of such a representation some additional notation has to be introduced.

Let \( \mathcal{U} = \mathbb{R}^m \) and for each \( u = (u_1, \ldots, u_k) \in \mathcal{U} \) denote

\[
d\zeta_j[u] = u_j, \quad j = 1, 2, \ldots, m, \quad d\zeta_0[u] = 1
\]

Denote the set \( \{0, 1, \ldots, m\} \) by \( \mathbb{Z}_m \). For each \( j_1, \ldots, j_k \in \mathbb{Z}_m \), \( k \geq 0, t \in T \), \( u \in \text{PC}(T, \mathcal{U}) \) define \( V_{j_1 \ldots j_k}[u](t) \in \mathbb{R} \) as

\[
V_{j_1 \ldots j_k}[u](t) = \begin{cases} 
1 & \text{if } k = 0 \\
\int_0^t d\zeta_{j_1}[u(\tau)]V_{j_1 \ldots j_{k-1}}[u(\tau)] d\tau & \text{if } k > 1
\end{cases}
\]

For each \( w_1, \ldots, w_k \in \mathbb{Z}_m, (t_1, \ldots, t_k) \in T^k \), \( u \in \text{PC}(T, \mathcal{U}) \) define

\[
V_{w_1 \ldots w_k}[u](t_1, \ldots, t_k) \in \mathbb{R}
\]

by

\[
V_{w_1 \ldots w_k}[u](t_1, \ldots, t_k) = V_{w_1}[t_1][u]V_{w_2}[t_2][\text{Shift}_1(u)] \cdots V_{w_k}[\text{Shift}_{k-1}(u)][t_k]
\]

where \( \text{Shift}_i(u) = \text{Shift}_{\sum_{j=1}^{i-1} t_j}(u), i = 1, 2, \ldots, k-1. \) We will call \( V_{w_1 \ldots w_k}[u](t_1, \ldots, t_k) \) the iterated integral of \( u \) at \( t_1, \ldots, t_k \) with respect to \( w_1, \ldots, w_k. \)

Let \( \mathcal{H} = (\mathcal{A}, \mathcal{U}, \mathcal{Y}, (X_q, A_q, \{B_q, j\})_{q \in \mathcal{Q}}, \{M_{q(\gamma, \gamma'), q} \mid q \in \mathcal{Q}, \gamma \in \Gamma\}) \) be a bilinear hybrid system. Notice that \( \text{Reach}(\mathcal{H}) \) \( \subseteq \mathcal{Q} \) and thus \( \text{Reach}(\mathcal{H}, \mathcal{H}_0) \) can be viewed as a subspace of \( \bigoplus_{q \in \mathcal{Q}} X_q \).

We will say that \( \mathcal{H} \) is semi-reachable from \( \mathcal{H}_0 \) if \( \bigoplus_{q \in \mathcal{Q}} X_q \) contains no proper vector subspace containing \( \text{Reach}(\mathcal{H}, \mathcal{H}_0) \) and the automaton \( \mathcal{A}_H \) is reachable from
If both bilinear hybrid morphisms are called bilinear hybrid isomorphisms. Two bilinear hybrid system realizations are isomorphic, such that

\[ H = (A, \mathcal{U}, \mathcal{V}, (Q, \mathcal{Y}, (X_q, A_q, B_{q,j})_{j=1,\ldots,m}, C_q)_{q \in Q}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in Q, \gamma \in \Gamma\}) \]

\[ H' = (A', \mathcal{U}', \mathcal{V}', (Q', \mathcal{Y}', (X_q', A_q', B_{q,j}')_{j=1,\ldots,m}, C_q')_{q \in Q'}, \{M_{\delta'(q,\gamma),\gamma,q} \mid q \in Q', \gamma \in \Gamma\}) \]

\[ A = (Q, \Gamma, O, \delta, \lambda) \quad \text{and} \quad A' = (Q', \Gamma, O, \delta', \lambda'). \]

A pair \( T = (T_D, T_C) \) is called a bilinear hybrid morphism from \((H, \mu)\) to \((H', \mu')\), denoted by \( T : (H, \mu) \to (H', \mu') \) if the following holds.

\[ T_D : (A, \mu_D) \to (A', \mu'_D) \]

where \( \mu_D(f) = \Pi_Q(\mu_D(f)), \mu'_D(f) = \Pi'_Q(\mu'_D(f)) \), is an automaton morphism and

\[ T_C : \bigoplus_{q \in Q} X_q \to \bigoplus_{q' \in Q'} X'_q \]

is a linear morphism, such that

(a) \( \forall q \in Q : T_C(X_q) \subseteq X'_{T_D(q)} \)

(b) \( T_C A_q = A'_{T_D(q)} T_C, T_C B_{q,j} = B'_{T_D(q),j} T_C, C_q = C'_{T_D(q)} T_C, \) for all \( q \in Q, j = 1, \ldots, m \).

(c) \( T_C M_{q_1,\gamma,q_2} = M'_{T_D(q_1),\gamma,T_D(q_2)} T_C, \) \( \forall q_1, q_2 \in Q, \gamma \in \Gamma, \delta(q_2,\gamma) = q_1 \).

(d) \( T_C(\Pi X_q(f)) = \Pi_{X'_{T_D(q)}}(\mu'(f)) \) for each \( q = \mu_D(f), f \in \Phi \).

The bilinear hybrid morphism \( T \) is said to be injective, surjective, or bijective if both \( T_D \) and \( T_C \) are respectively injective, surjective, or bijective. Bijective bilinear hybrid morphisms are called bilinear hybrid isomorphisms. Two bilinear hybrid system realizations are isomorphic if there exists a bilinear hybrid isomorphism between them.

Notice that the set \( \bigcup_{q \in Q} X_q \) can be naturally viewed as a subset of \( \bigoplus_{q \in Q} X_q \). It is easy to see that the map \( T_C : \bigoplus_{q \in Q} X_q \to \bigoplus_{q' \in Q'} X'_q \) is completely determined by its restriction to \( \bigcup_{q \in Q} X_q \). We will denote this restriction by \( M(T) \).

Recall the concept of hybrid system morphism from Section 6. The following proposition clarifies the relationship between morphisms of bilinear hybrid systems and hybrid system morphisms.

**Proposition 18.** If the pair \( T = (T_D, T_C) \) defines a bilinear hybrid morphism \( T : (H_1, \mu_1) \to (H_2, \mu_2) \), then \( \psi(T) = (T_D, M(T)) \) defines a hybrid system morphism \( H(T) : (H_1, \mu_1) \to (H_2, \mu_2) \) in sense of Section 6. Moreover, \( H(T) \) is a hybrid isomorphism if and only if \( T \) is a bilinear hybrid isomorphism.
8.2 Input-output maps of bilinear hybrid systems

This subsection reviews the notion of hybrid Fliess-series expansion and its connection to input-output maps of bilinear hybrid systems.

Let \( \Gamma = \Gamma \cup Z_m \). Then any \( w \in \Gamma^* \) is of the form \( w = w_1\gamma_1 \cdots w_k\gamma_kw_{k+1}, \gamma_1, \ldots, \gamma_k \in \Gamma, \ w_1, \ldots, w_{k+1} \in Z_m^*, \ k \geq 0 \). A map \( c : \Gamma^* \to \mathcal{Y} \) is called a hybrid generating convergent series on \( \Gamma^* \) if there exists \( K, M > 0, K, M \in \mathbb{R} \) such that for each \( w \in \Gamma^* \),

\[
||c(w)|| < KM^{|w|}
\]

where \( ||.|| \) is some norm in \( \mathcal{Y} = \mathbb{R}^p \). The notion of generating convergent series is related to the notion of convergent power series from [7]. Let \( c : \Gamma^* \to \mathcal{Y} \) be a hybrid generating convergent series. For each \( u \in PC(T, \mathcal{U}) \) and \( s = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*, t_{k+1} \in T \) define the series

\[
F_c(u, s, t_{k+1}) = \sum_{w_1, \ldots, w_{k+1} \in Z_m^*} c(w_1\gamma_1 \cdots \gamma_kw_{k+1}) V_{w_1, \ldots, w_{k+1}}[u](t_1, \ldots, t_{k+1})
\]

Lemma 13. Let \( c : \Gamma^* \to \mathcal{Y} \) be a hybrid generating convergent series. Then for each \( u \in PC(T, \mathcal{U}) \), \( w \in (\Gamma \times T)^* \), \( t \in T \), the series \( F_c(u, w, t) \) is absolutely convergent. Thus, the map

\[
F_c : PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T \ni (u, w, t) \mapsto F_c(u, w, t) \in \mathcal{Y}
\]

is well-defined. The hybrid convergent generating series \( c \) determines the map \( F_c \) uniquely, that is, if for some hybrid convergent generating series \( d \) \( F_c = F_d \), then \( c = d \).

Now we are ready to define the concept of hybrid Fliess-series representation of a set of input/output maps, which is related to the concept of Fliess-series expansion in [7]. For any map \( f \in F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times \mathcal{O}) \), define \( f_C = \Pi_\mathcal{Y} \circ f \), \( f_D = \Pi_\mathcal{O} \circ f \). Let \( \Phi \subseteq F(PC(T, \mathcal{U}) \times (\Gamma \times T)^* \times T, \mathcal{Y} \times \mathcal{O}) \).

Definition 6 (Hybrid Fliess-series expansion). \( \Phi \) is said to admit a hybrid Fliess-series expansion if

1. For each \( f \in \Phi \) there exists a generating convergent series \( c_f : \Gamma^* \to \mathcal{Y} \) such that \( F_{c_f} = f_C \)

2. For each \( f \in \Phi \) the map \( f_D \) depends only on \( \Gamma^* \), that is, for each \( w \in \Gamma^* \),

\[
\forall u_1, u_2 \in PC(T, \mathcal{U}), \tau_1, \tau_2 \in T^{|w|}, t_1, t_2 \in T : f_D(u_1, (w, \tau_1), t_1) = f_D(u_2, (w, \tau_2), t_2)
\]

We will regard \( f_D \) as a function \( f_D : \Gamma^* \to \mathcal{O} \).

The notion of hybrid Fliess-series representation is an extension of the notion of Fliess-series for input-output maps of non-linear systems, see [7]. The following proposition gives a description of the hybrid Fliess-series expansion of \( \Phi \) in the case when \( \Phi \) is realized by a bilinear hybrid system.
Proposition 19. \((H, \mu)\) is a bilinear hybrid system realization of \(\Phi\) if and only if \(\Phi\) has a hybrid Fliess-series expansion such that for each \(f \in \Phi\), \(w_1 \gamma_1 \cdots \gamma_k w_{k+1} \in \Gamma^*\), \(\gamma_1, \ldots, \gamma_k \in \Gamma\), \(w_1, \ldots, w_{k+1} \in \mathbb{Z}_m^*\), \(k \geq 0\)

\[
\begin{align*}
    c_f(w_1 \gamma_1 \cdots \gamma_k w_{k+1}) &= C_{q_k} B_{q_k, w_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_{0}} B_{q_0, w_1} \\
    f_D(\gamma_1 \cdots \gamma_k) &= \lambda(q_0, \gamma_1 \cdots \gamma_k)
\end{align*}
\]

(14)

where \(\mu(f) = (q_0, \mu_C(f))\) and \(q_i = \delta(q_0, \gamma_1 \cdots \gamma_i), i = 0, \ldots, k\).

8.3 Realization of input-output maps by bilinear hybrid systems

In this section the solution to the realization problem for bilinear hybrid systems will be presented. In addition, characterisation of minimal bilinear hybrid systems realizing the specified set of input-output maps will be given. We will use the theory of hybrid formal power series developed in Section 5.

Let us recall the characterisation of semi-reachability and observability for bilinear hybrid systems presented in [9, 12]. Using the notation of Definition 4, the following holds.

Theorem 19. The bilinear hybrid system \(H\) is observable if and only if

(i) \(A_H = A\) is observable, and

(ii) For each \(q \in Q\),

\[
O_{H,q} = \bigcap_{\gamma_1, \ldots, \gamma_k \in \Gamma} \bigcap_{k \geq 0} \ker C_{q_k} B_{q_k, w_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} B_{q_0, w_1} = \{0\}
\]

where \(q_i = \delta(q_0, \gamma_1 \cdots \gamma_i), 0 \leq i \leq k, k \geq 0, q = q_0\).

Notice that part (i) of the theorem above is equivalent to

\[
v_H((q_1, 0), \cdot) = v_H(q_2, 0), \cdot \iff q_1 = q_1, \forall q_1, q_2 \in Q
\]

Part (ii) of the theorem says that for each \(q \in Q\):

\[
v_H((q, x_1), \cdot) = v_H((q, x_2), \cdot) \iff x_1 = x_2, \forall x_1, x_2 \in X_q
\]

Theorem 20. \((H, \mu)\) is semi-reachable if and only if \((A_H, \mu_D), \mu_D = \Pi_Q \circ \mu\), is reachable and \(\dim W_H = \sum_{q \in Q} \dim X_q\), where

\[
W_H = \text{Span}\{B_{q_k, w_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_{0}} B_{q_0, w_1} x_f, | (q_f, x_f) = \mu(f), f \in \Phi, w_1, \ldots, w_{k+1} \in \mathbb{Z}_m^*, q_j = \delta(q_f, \gamma_1 \cdots \gamma_j), 0 \leq j \leq k, k \geq 0\}
\]

Let \(\Phi \subseteq F(PC(T, U) \times (\Gamma \times T)^* \times T, Y \times O)\) be a set of input-output maps. Assume that \(\Phi\) has a hybrid Fliess-series expansion. Then Proposition 19 allows us to reformulate the realization problem in terms of rationality of certain hybrid
formal power series. Recall that $\bar{\Gamma} = \Gamma \cup \mathbb{Z}_m$. Let $J = \Phi$ and for each $f \in \Phi$ define the hybrid formal power series $T_f \in \mathbb{R}^p \prec \bar{\Gamma}^* \rhd \times F(\Gamma, O)$ by

$$(T_f)_C = e_f \text{ and } (T_f)_D = f_D$$

It is easy to see that $J$ is a hybrid power series index set with $J_1 = J = \Phi$ and $J_2 = \emptyset$. Define the indexed set of hybrid formal power series associated with $\Phi$ by

$$\Psi_\Phi = \{ T_f \in \mathbb{R}^p \prec \bar{\Gamma}^* \rhd \times F(\Gamma^*, O) \mid f \in \Phi \}$$

It is easy to see that $\Psi_\Phi$ is a well-posed indexed set of hybrid formal power series with the index set $J$. Define the Hankel-matrix $H_\Phi$ of $\Phi$ as $H_\Phi = H_{\Psi_\Phi}$. Notice that if $\Phi$ is finite, then $\Psi_\Phi$ is a finite set. Let

$$H = (A, U, Y, (X_q, A_q, \{B_{q,j}\})_{j=1,\ldots,m}, C_q)_{q \in \Phi}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in \Phi, \gamma \in \Gamma \}$$

$(H, \mu)$ be a bilinear hybrid system realization with $\mu : \Phi \to \bigcup_{q \in \Phi} \{q\} \times X_q$. Define the hybrid representation $H_{HR,\mu}$ associated with $(H, \mu)$ by

$$H_{HR,\mu} = (A, (X_q, \{A_{q,z}\})_{z \in X_1}, C_q)_{q \in \Phi}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in \Phi, y \in X_2 \}, J, \mu)$$

where $J = J_1 = \Phi, J_2 = \emptyset, X = \bar{\Gamma}, X_1 = \mathbb{Z}_m, X_2 = \Gamma$ and for each $q \in \Phi, j = 1,\ldots,m$,

$$A_{q,0} = A_q \text{ and } A_{q,j} = B_{q,j}$$

Conversely, let $HR = (A, (X_q, \{A_{q,z}\})_{z \in X_1}, C_q)_{q \in \Phi}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in \Phi, y \in X_2 \}, J, \mu)$ be a hybrid representation with index set $J = \Phi$ such that $X_1 = \mathbb{Z}_m, X_2 = \Gamma, J_1 = \Phi, J_2 = \emptyset, X = \bar{\Gamma}$. Define the bilinear hybrid realization $(H_{HR}, \mu_{HR})$ associated with $HR$ as follows

$$H_{HR} = (A, U, Y, (X_q, A_q, \{B_{q,j}\})_{j=1,\ldots,m}, C_q)_{q \in \Phi}, \{M_{\delta(q,\gamma),\gamma,q} \mid q \in \Phi, \gamma \in \Gamma \}$$

and $\mu_{HR} = \mu$, where for each $q \in \Phi, j = 1,\ldots,m$,

$$A_q = A_{q,0} \text{ and } B_{q,j} = A_{q,j}$$

It is easy to see that $(H_{HR_{\mu,\mu}}, \mu_{HR_{\mu,\mu}}) = (H, \mu)$ and $H_{HR_{\mu,\mu}} = HR$ for any hybrid representation $HR$ and bilinear hybrid realization $(H, \mu)$. It is also easy to see that $\dim H = \dim H_{HR,\mu}$.

The following theorem follows easily from Proposition 19 and plays a crucial role in realization theory of bilinear hybrid system.

**Theorem 21.** A bilinear hybrid system $(H, \mu)$ is a realization of $\Phi$ if and only if $\Phi$ has a hybrid Fliess-series expansion and $H_{HR,\mu}$ is a hybrid representation of $\Psi_\Phi$. Conversely, if $\Phi$ has a hybrid Fliess-series expansion and $HR$ is a hybrid representation of $\Psi_\Phi$, then $(H_{HR}, \mu_{HR})$ is a bilinear hybrid system realization of $\Phi$.

The theorem above allows us to reduce the realization problem for bilinear hybrid systems to existence of a hybrid representation of a indexed set of hybrid formal power series. Moreover, Theorem 19 and Theorem 20 allow us to relate observability and semi-reachability of bilinear hybrid systems to observability and reachability of hybrid representations.
Theorem 22. A bilinear hybrid system realization $(H, \mu)$ is observable if and only if $HR_{H, \mu}$ is observable. A bilinear hybrid system realization $(H, \mu)$ is semi-reachable if and only if $HR_{H, \mu}$ is reachable.

Notice that both $H$ and $HR_{H, \mu}$ have the same state-space. It is easy to see that the following holds.

Lemma 14. Let $(H_i, \mu_i), i = 1, 2$ be two bilinear hybrid systems. If $T : (H_1, \mu_1) \to (H_2, \mu_2)$ is a bilinear hybrid morphism, then $T$ is also a $T : HR_{H_1, \mu_1} \to HR_{H_2, \mu_2}$ hybrid representation morphism. Conversely, if $HR_i, i = 1, 2$ are two hybrid representations with hybrid power series index set $J = \Phi$ and $T : HR_1 \to HR_2$ is a a hybrid representation morphism then $T$ can be viewed as a $T : (HR_{H_1, \mu_{HR_1}}) \to (HR_{H_2, \mu_{HR_2}})$ bilinear hybrid morphism. The map $T$ is a surjective, injective, isomorphism as a hybrid bilinear morphism if and only if $T$ is surjective, injective, isomorphism as a hybrid representation morphism.

Let $\Phi_D = (\Psi_\Phi)_D$. From the discussion above, using the results on theory of hybrid formal power series (Theorem 9 and Theorem 10 and Corollary 3) we can derive the following theorem, which was already published in [9].

Theorem 23 (Realization of input/output map). Let $\Phi \subseteq F(\mathcal{P}(T, U) \times (\Gamma \times T^*) \times T, Y \times O)$ be a set of input-output maps. The following are equivalent.

(i) $\Phi$ has a realization by a bilinear hybrid system,

(ii) $\Phi$ has a hybrid Fliess-series expansion, $\Psi_\Phi$ is rational indexed set of hybrid formal power series

(iii) $\Phi$ has a hybrid Fliess-series expansion, $\text{rank } H_\Phi < +\infty$ and $\Phi_D$ has a realization by a finite Moore-automaton, i.e. $\text{card}(W_{\Phi_D}) < +\infty$.

Below we will give a characterisation of minimal bilinear hybrid systems.

Theorem 24 (Minimal realization). If $\Phi$ has a bilinear hybrid system realization, then $\Phi$ has a minimal bilinear hybrid system realization. If $(H, \mu)$ is a bilinear hybrid system realization of $\Phi$, then the following are equivalent.

(i) $(H, \mu)$ is minimal,

(ii) $(H, \mu)$ is semi-reachable and it is observable,

(iii) For each $(H', \mu')$ semi-reachable bilinear hybrid realization of $\Phi$ there exists a surjective bilinear hybrid morphism $T : (H', \mu') \to (H, \mu)$. In particular, all minimal hybrid bilinear systems realizing $\Phi$ are isomorphic.

The theory of hybrid formal power series developed in Section 5 allows us to formulate a partial realization theorem for bilinear hybrid systems. It also enables us to formulate algorithms for deciding observability and semi-reachability of bilinear hybrid systems and to give an algorithm for constructing a minimal bilinear hybrid system realization based on a specified hybrid system realization. In fact, the results presented below are more general than the ones described in [9]. Notice that the algorithmic aspects of realization theory are treated in this paper in a much more detailed manner than in [9].

Let $\Phi$ be a collection of input-output maps and assume that $\Phi$ admits a hybrid Fliess-series expansion. It is easy to see that all information needed
for constructing the indexed set of hybrid formal power series $\Omega = \Psi_\Phi$ can be obtained (in theory) from the set of input-output maps $\Phi$, more precisely, from the generating series $c_f$ and discrete input-output maps $f_D$ for all $f \in \Phi$. In fact, the values of $c_f$ can be recovered from $f$ by taking high-order derivatives with respect to time and continuous inputs.

Assume that $\Phi$ is a finite collection of input-output maps. Notice that it also implies that the index set $J = \Phi$ of $\Psi_\Phi$ is finite. Unless stated otherwise, we will use this finiteness assumption in the rest of this section.

Our first goal is to construct a bilinear hybrid realization of $\Phi$ from finite number of data points. Recall the results of Subsection 5.4. It is easy to see that if $\Phi$ is finite then all the data for constructing $W_{\Omega,N,D,D}$ and $H_{\Omega,N,N}$ can be obtained from the input-output maps of $\Phi$ and the number of data points needed for constructing $W_{\Omega,N,D,D}$ and $H_{\Omega,N,N}$ is finite. Theorem 11 yields that the finite data from $W_{\Omega,N,D,D}$ and $H_{\Omega,N,N}$ can be used to compute a minimal hybrid representation of $\Psi_\Phi$. But any minimal hybrid representation $HR$ of $\Omega$ yields a minimal bilinear hybrid realization $(H_{HR}, \mu_{HR})$ of $\Phi$. Thus, we get the following result. Denote $H_{\Phi,N,M} = H_{\Psi_\Phi,N,M}, \mathcal{D}_{\Phi,N} = \mathcal{D}_{\Psi_\Phi,N}$.

**Theorem 25.** Assume that $\Phi$ is a finite collection of input-output maps and $\Phi$ admits a hybrid Fliess series expansion. Assume that rank $H_{\Phi,N,N} = \text{rank } H_\Phi$ and $\text{card}(W_{\Phi,N,D,D}) = \text{card}(W_{\Phi,N})$. Let $HR_{N,D}$ be the hybrid representation from Theorem 11. Then $(H_{N,D,N,D}) = (H_{HR_{N,D}}, \mu_{HR_{N,D}})$ is a minimal bilinear hybrid system realization of $\Phi$ and it can be constructed from finite data which can be obtained directly from $\Phi$. In particular, if $\Phi$ has a bilinear hybrid system realization $(H, \mu)$ such that $\dim H = (p, q)$ and $\max\{p, q\} \leq N$, then $(H_{N,N,N,N})$ is a minimal bilinear hybrid system realization of $\Phi$ and it can be constructed from finitely many data which is directly obtainable from $\Phi$.

The results of Subsection 5.4 also allow us to check observability and semi-reachability of bilinear hybrid systems algorithmically. Indeed, consider a bilinear hybrid system realization $(H, \mu)$. It is easy to see that the construction of $HR_{H,\mu}$ can be carried out by a computer algorithm. It follows that $HR_{H,\mu}$ is reachable if and only if $(H, \mu)$ is semi-reachable and $HR_{H,\mu}$ is observable if and only if $H$ is observable. Recall the procedures $\text{IsHybRepObservable}$ and $\text{IsHybRepReachable}$. To check semi-reachability of $(H, \mu)$ we can apply $\text{IsHybRepReachable}$ to $HR_{H,\mu}$. To check observability of $(H, \mu)$ we can apply $\text{IsHybRepObservable}$ to $HR_{H,\mu}$. Finally, we can apply $\text{ComputeMinimalHybRep}$ to $HR_{H,\mu}$ to obtain a minimal hybrid representation $HR$ and then we can construct $(H_{HR}, \mu_{HR})$ which will be a minimal bilinear hybrid system realization of $\Phi$. Notice that the construction of $(H_{HR}, \mu_{HR})$ can be carried out algorithmically. Thus, if all the entries of the system matrices of $H$ are rational and all the values of $\mu$ are rational, then observability and semi-reachability of $(H, \mu)$ is algorithmically decidable and a minimal bilinear hybrid realization of $\Phi$ can be constructed from $(H, \mu)$ by an algorithm in sense of classical Turing computability.

Below we will present a numerical example

**Example**

Consider the following bilinear hybrid system. Consider the Moore-automaton $A = (Q, \Gamma, O, \delta, \lambda)$, where $Q = \{q_1, q_2\}$, $\Gamma = \{a, b\}$ and $O = \{0\}$. Define the discrete state transition map by $\delta(q_1, a) = q_1, \delta(q_1, b) = q_2, \delta(q_2, b) = q_2, \delta(q_2, a) = q_2$. Define the readout map $\lambda(q_1) = \lambda(q_2) = 0$. Consider the linear hybrid
system

\[ H = (A, U, \mathcal{Y}, (X_q, A_q, \{B_q\}_j), \{C_q\}_q)_{q \in \mathbb{Q}} \]  

where \( \mathcal{Y} = \mathbb{R} \), i.e. \( p = m = 1 \), \( X_{q_1} = \mathbb{R}^3 \) and \( X_{q_2} = \mathbb{R}^2 \) and the matrices \( A_q, B_{q,1}, C_q, q \in \{q_1, q_2\} \) are of the following form

\[
A_{q_1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad B_{q_1,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C_{q_1} = [1, 1, 1]
\]

\[
A_{q_2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B_{q_2,1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C_{q_2} = [1, 1]
\]

The linear reset maps are of the following form

\[
M_{q_1,a,q_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_{q_2,b,q_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

The input/output map \( v_H((q_2, x_0), \cdot) \) induced by \((q_2, x_0), x_0 = [0, 1]^T\), is quite complex, as a demonstration we will present below the output to the discrete input sequence \((b, t_1)(a, t_2)(a, t_3)(b, t_4)\).

\[
v((q_2, x_0), u, (b, t_1)(a, t_2)(a, t_3)(b, t_4), t_5) = \\
\left( o, \sum_{w_1, \ldots, w_5 \in \mathbb{Z}_m^*} 3^{n_z(w_1) + n_z(w_2)} V_{w_1, \ldots, w_5}[u](t_1, \ldots, t_5) \right)
\]

where \( n_z(w) \) is the number of occurrences of the symbol 0 in \( w \), \( V_{w_1, \ldots, w_5}[u](t_1, \ldots, t_5) \) - product of iterated integrals.

A minimal realization of \( v_H((q_2, x_0), \cdot) \) of the following form.

\[
H_m = (A^m, U, \mathcal{Y}, (X_q^m, A_q^m, \{B_q^m\}_j), \{C_q^m\}_q)_{q \in \mathbb{Q}_m^m, \{M_{q_1,a,q_2}^m\}_q} \]  

where \( \mathcal{Y} = \mathbb{R}, Q^m = \{q\}, X^m_q = \mathbb{R}^2 \), the automaton \( A^m = (Q^m, \Gamma, O, \delta^m, \lambda^m) \) is given by

\[
\delta^m(q, z) = q, z \in \{a, b\} \quad \text{and} \quad \lambda^m(q) = o
\]

The matrices \( A_q^m, B_{q,1}^m, C_q^m, M_{q_1,a,q_2}^m, m \in \{a, b\} \)

\[
A_q^m = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad B_{q,1}^m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_q^m = [1, -1]
\]

Reset maps:

\[
M_{q,a,q_2}^m = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \quad M_{q,a,a}^m = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}
\]

Define \( \mu_m(v_H((q_2, x_0), \cdot)) = (q, z_0) \) by \( z_0 = [0, -1]^T \). Then \( (H_m, \mu_m) \) is a minimal bilinear hybrid system realization of \( v_H((q_2, x_0), \cdot) \). The realization \( H_m \) was computed using a Matlab implementation of the algorithm presented in the paper.
9  Conclusions
The abstract framework of hybrid formal power series was presented. Application of the new theory to realization theory of linear and bilinear systems was discussed. Further research is directed towards extending the scope of application of rational hybrid formal power series.

Acknowledgment
The author thanks Jan H. van Schuppen, Pieter Collins and Luc Habets for useful discussions and suggestions. Part of this paper was written while the author stayed at INRIA Sophia-Antipolis as a CTS Fellow HPMT-GH-01-00278-158.

References


