Reachability of Linear Switched Systems: 
Differential Geometric Approach

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Abstract

The paper investigates the structure of the reachable set of linear switched systems. The structure of the reachable set is determined using techniques from classical nonlinear systems theory, namely, the theory of orbits developed by H. Sussman and the realization theory for nonlinear systems developed by B. Jakubczyk.

Key words: Hybrid systems, switched linear systems, reachable set
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1 Introduction

Linear switched systems are a popular and well studied subclass of switched systems. Large number of works have been published on the topic, for a comprehensive survey see [1].

This paper deals with the reachability and the structure of the reachable set of linear switched systems. The issue of reachability for linear switched systems has been addressed in a number of papers, see [2,3]. An exhaustive study of the reachability of linear switched systems is presented in [2]. On the level of results the current paper doesn’t offer anything more than [2]. The novelty lies in the methods which are used to prove these results. Namely, the current paper uses techniques from differential geometric theory of nonlinear systems theory to derive the structure of the reachable set. The main tool is the theory of orbits, developed by H. Sussman in [4], and realization theory for nonlinear systems.
systems by B. Jakubczyk [5]. The theory of orbits allows one to compute the structure of the set of states which are weakly reachable, i.e. reachable in positive or negative time from zero. This, in turn, allows the application of the classical nonlinear conditions for accessibility to the system restricted to the set of the weakly reachable states. Accessibility of the restricted system and the linear structure of the weakly reachable set makes it easy to determine the structure of the reachable set.

In the author’s opinion, the proof of the paper is more conceptual and it makes the connection between the classical systems theory and the theory of hybrid systems more transparent. The author also hopes that the methods employed in the paper can be extended to more general classes of hybrid systems.

The outline of the paper is the following. Section 2 gives the precise mathematical formulation of concepts and problems which are dealt with in this paper. Some elementary properties of switched systems are also presented in Section 2. This section also contains the statement of the main result. Section 3 contains the results from classical nonlinear systems theory, which are needed for the proof of the main result. Section 4 contains the proof main result of the paper. The paper contains most of the results on nonlinear systems theory and differential geometry needed to derive the main results. Nevertheless some basic knowledge of these subjects is necessary to follow all the details. Good references on these topics are [6,7].

2 Problem formulation

This sections contains the definition and some elementary properties of switched systems. At the end of the section the main theorem of the paper is formulated.

**Definition 1** A switched (control) system is a tuple

\[ \Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0) \]

where

- \( T = \mathbb{R}_+ \)
- \( \mathcal{X} = \mathbb{R}^n \) is the state-space
- \( \mathcal{Y} = \mathbb{R}^p \) is the output-space
- \( \mathcal{U} = \mathbb{R}^m \) is the input-space
- \( Q \) is the finite set of discrete modes
- \( f_\sigma(x, u) \) is a smooth function and it is globally Lipschitz in \( x \).
- \( h_\sigma : \mathcal{X} \to \mathcal{Y} \) is smooth map for each \( \sigma \in Q \)
- \( x_0 \in \mathcal{X} \) is the initial state.
For sets $A, B,$ denote by $PC(A, B)$ the class of piecewise-continuous mappings from $A$ to $B$. For a set $A$ denote by $A^*$ the set of finite strings of elements of $A$. For $w = a_1a_2\cdots a_k \in A^*$ the length of $w$ is denoted by $|w|$, i.e. $|w| = k$. The empty sequence is denoted by $\epsilon$. The length of $\epsilon$ is zero: $|\epsilon| = 0$. For any function $f : T \to A$ and any $t \in T$ define $\text{Shift}_tf$ by $(\text{Shift}_tf)(s) = f(s + t)$. If $A, B$ are two sets, then the set $(A \times B)^*$ will be identified with the set $\{(u, w) \in A^* \times B^* \mid |u| = |w|\}$. If $\cdot : A \times B \to B$ is an arbitrary function, then for all $w = w_1w_2\cdots w_k \in B^*$, $a \in A$ define $a \cdot w := (a \cdot w_1)(a \cdot w_2)\cdots (a \cdot w_k) \in B^*$. Let $D$ be a set. The relation $R \subseteq D^* \times D^*$ is called a congruence relation if $R$ is an equivalence relation and $\forall w, v, u, s \in D^*: (w, v) \in R \implies (uws, uvv) \in R$.

The inputs of the switched system $\Sigma$ are the functions $PC(T, U)$ and the sequences $(Q \times T)^*$. That is, the switching sequences are part of the input, it is given externally and we allow any switching sequence to occur. Let $u \in PC(T, U)$ and $w = (q_1, t_1)(q_2, t_2)\cdots (q_k, t_k) \in (Q \times T)^*$. The inputs $u$ and $w$ steer the system $\Sigma$ from state $x_0$ to state $x_{\Sigma}(x_0, u, w)$ given by

$$x_{\Sigma}(x_0, u, w) = F(q_k, \text{Shift}_{\sum_{i=1}^{k-1} t_i} u, t_k) \circ F(q_{k-1}, \text{Shift}_{\sum_{i=2}^{k-1} t_i} u, t_{k-1}) \circ \cdots \circ F(q_1, u, t_1)(x_0)$$

where $F(q, u, t) : \mathcal{X} \to \mathcal{X}$ and for each $x \in \mathcal{X}$ the function $F(q, u, t, x) : t \mapsto F(q, u, t)(x)$ is the solution of the differential equation

$$\frac{d}{dt}F(q, u, t, x) = f_q(F(q, u, t, x), u(t)), \quad F(q, u, 0, x) = x$$

The empty sequence $\epsilon \in (Q \times T)^*$ leaves the state intact: $x_{\Sigma}(x_0, u, \epsilon) = x_0$.

Whenever it doesn’t create confusion, we will use the notation $x(x_0, ..)$ instead of $x_{\Sigma}(x_0, ..)$.

The reachable set of a system $\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$ is defined by

$$\text{Reach}(\Sigma) = \{x(x_0, u, w) \in \mathcal{X} \mid u \in PC(T, U), w \in (Q \times T)^*\}$$

Denote by $PC_{\text{const}}(T, U)$ the set of piecewise-constant input functions. A function $u(.) : T \to \mathcal{U}$ is called piecewise-constant if for each $[t_0, t_k] \subseteq T$ there exist $t_0 < t_1 < \cdots t_k$ such that $u|_{[t_i, t_{i+1}]}$ is constant for all $i = 0\ldots k - 1$. It is well-known that for each $u(.) \in PC(T, U)$ there exists a sequence $u_n(.) \in PC_{\text{const}}(T, U), n \in \mathbb{N}$ such that $\lim_{n \to +\infty} u_n(.) = u(.)$ pointwise. Given a switched system $\Sigma$, by continuity of solutions of differential equations we get that
∀x ∈ X : ∀w ∈ (Q × T)*, ∀u(.) ∈ PC(T, U), ∀u_n(.) ∈ PC_{const}(T, U) :
\lim_{n \to \infty} u_n(.) = u(.) \implies \lim_{n \to \infty} x(x, u_n(.), w) = x(x, u(.), w) \quad (1)

The set of states reachable by piecewise-constant input is defined as

\[ \text{Reach}_{\text{const}}(\Sigma) = \{ x(x_0, u, w) \in X \mid w \in (Q \times T)^*, u(.) \in PC_{\text{const}}(T, U) \} \]

From (1) one gets immediately following proposition

**Proposition 2** Given a switched system \( \Sigma \), the set of states reachable by piecewise-constant input is dense in the set \( \text{Reach}(\Sigma) \), i.e.

\[ \text{Cl}(\text{Reach}_{\text{const}}(\Sigma)) = \text{Reach}(\Sigma) \]

For any \( u \in PC(T, U) \), \( w, v \in (Q \times T)^* \) it holds that \( x(x_0, u, w(q, t)(q', t')v) = x(x_0, u, w(q, t + t')v) \). Define \( R \subseteq (Q \times T)^* \times (Q \times T)^* \) by \( w(q, t)(q', t')v \Rightarrow w(q, t + t')v \) and let \( R^* \) be the smallest equivalence relation containing \( R \).

**Proposition 3** For any \( u \in PC_{\text{const}}(T, U) \) and \( w \in (Q \times T)^* \) there exists \( w' = (q_1, t_1) \cdots (q_k, t_k) \), \( w' R^*w \) such that \( \forall i = 1, 2, \ldots, k \) the function \( u|_{[\sum_{i=1}^{k-1} t_j, \sum_{i=1}^{k} t_j]} \) is a constant.

It is clear that for any \( w = (q_1, t_1)(q_2, t_2) \cdots (q_k, t_k) \in (Q \times T)^* \) the value \( x(x_0, u(.), w) \) depends on \( u(.)|_{[0, \sum_{i=1}^{k} t_i]} \). Proposition 2 and Proposition 3 imply that without loss of generality it is enough to consider pairs \( (w, u) \) where \( w = (q_1, t_1) \cdots (q_k, t_k) \in (Q \times T)^* \) and \( u \in PC([0, \sum_{i=1}^{k} t_i], U), u|_{[\sum_{i=1}^{k-1} t_j, \sum_{i=1}^{k} t_j]} = u_i \in U \) for \( i = 1, 2, \ldots, k \).

In the sequel we will use the following abuse of notation. For each \( x \in X, u \in U^* \), \( w \in Q^* \) and \( \tau \in T^* \) such that \( |t| = |w| = |u| \) we define

\[ x(x, u, w, \tau) := x(x, \tilde{u}, (w_1, t_1)(w_2, t_2) \cdots (w_k, t_k)) \]

where \( \tilde{u}|_{[\sum_{i=1}^{j-1} t_j, \sum_{i=1}^{j} t_i]} = u_j \) for \( j = 1, 2, \ldots, k \), and \( \tilde{u}|_{[\sum_{i=1}^{k} t_i, +\infty)} \) is arbitrary.

\( x(x_0, \ldots, \ldots) \) will be considered as function with its domain in \( (U \times Q \times T)^* \) or equivalently in \( \{(u, w, \tau) \in U^* \times Q^* \times T^* \mid |u| = |w| = |\tau|\} \). It is easy to see that

\[ \text{Reach}_{\text{const}}(\Sigma) = \{ x(x_0, u, w, \tau) \mid (u, w, \tau) \in (U \times Q \times T)^* \} \]

**Definition 4 (Linear switched systems)** A switched system \( \Sigma \) is called linear, if \( x_0 = 0 \) and for each \( q \in Q \) there exist linear mappings \( A_q : X \to X \), \( B_q : U \to X \) and \( C_q : X \to Y \) such that

- \( \forall u \in U, \forall x \in X : f_q(x, u) = A_q x + B_q u \)
• $\forall x \in X : h_q(x) = C_q x$

To make the notation simpler, linear switched systems will be denoted by $\Sigma = (X, U, Y, Q; \{(A_q, B_q, C_q) \mid q \in Q\})$

Notice that the initial state of a linear switched system is zero.

Below some elementary properties of linear switched systems are presented. The results are elementary, and listed for reference only.

**Proposition 5** Consider a linear switched system $\Sigma$. Then the following holds

1. $\forall u(.) \in PC(T, U), x_0 \in X, w = (q_1, t_1)(q_2, t_2) \cdots (q_k, t_k) \in (Q \times T)^*$

$$x(x_0, u(.), w) = \exp(A_{q_1} t_k) \exp(A_{q_{k-1}} t_{k-1}) \cdots \exp(A_{q_2} t_1)x_0 + \int_0^{t_k} \exp(A_{q_k}(t_k - s))B_{q_k} u(\sum_{i=1}^{k-1} t_i + s)ds + \exp(A_{q_k} t_k) \int_0^{t_{k-1}} \exp(A_{q_{k-1}}(t_{k-1} - s))B_{q_{k-1}} u(\sum_{i=1}^{k-2} t_i + s)ds + \cdots \exp(A_{q_k} t_k) \exp(A_{q_{k-1}} t_{k-1}) \cdots \exp(A_{q_2} t_1) \int_0^{t_1} \exp(A_{q_1}(t_1 - s))B_{q_1} u(s)ds$$

2. $\forall w \in Q^*, u, v \in U^*, \tau \in T^*, \forall \alpha, \beta \in \mathbb{R}$

$$x(x_0, \alpha u + \beta v, w, \tau) = \alpha x(x_0, u, w, \tau) + \beta x(x_0, v, w, \tau)$$

3. $\forall u, v \in U^*, w, p \in Q^*, \tau_1, \tau_2 \in T^*, p = p_1 p_2 \cdots p_k, \tau_2 = t_1 t_2 \cdots t_k$

$$x(0, u, v, wp, \tau_1 \tau_2) = x(0, v, p, \tau_2) + x(0, u 00 \cdots 0, wp, \tau_1 \tau_2) \text{ [}p\text{-times]}$$

$$x(0, u 00 \cdots 0, wp, \tau_1 \tau_2) = \exp(A_{p_k} t_k) \cdots \exp(A_{p_1} t_1) x(0, u, w, \tau_1) \text{ [}p\text{-times]}$$

The main result of the paper is the following.

**Theorem 6** Consider a switched linear system $\Sigma = (X, U, Y, Q; \{(A_q, B_q, C_q) \mid q \in Q\})$.

(a) $\text{Reach}(\Sigma) = \{A^{j_1}_{q_1} A^{j_2}_{q_2} \cdots A^{j_k}_{q_k} B_z u \mid q_1, q_2, \ldots q_k, z \in Q, j_1, j_2, \ldots j_k \geq 0, u \in U\}$

(b) There exists a switching sequence $w \in (Q \times T)^+$ such that $\text{Reach}(\Sigma) = \{x(0, u, w) \mid u \in PC(T, U)\}$
3 Preliminaries on nonlinear systems theory

Below the results of [4–6] will be reviewed. Basic knowledge of differential geometry is assumed. For references see [7]. In the sequel, unless stated otherwise, by manifold we mean a smooth finite-dimensional manifold, i.e. a topological space, which is Hausdorff space, second countable and locally homeomorphic to open subsets of \( \mathbb{R}^n \), and is endowed with a smooth (analytic) differentiable structure. Let \( M \) be a manifold. Then for each \( x \in M \) the tangent space of \( M \) at \( x \) will be denoted by \( T_x M \), the tangent bundle of \( M \) will be denoted by \( TM = \bigcup T_x M \). Let \( X \) be a vector field of \( M \). Then \( X^t(x) \) denotes the flow of \( X \) passing through the point \( x \) at time \( t \). The mapping \( D : M \rightarrow 2^{TM} \) is called a distribution if for each \( x \in M \), \( D(x) \) is a subspace of \( T_x M \). A sub-manifold \( N \) of \( M \) is an integral sub-manifold of the distribution \( D \) if for each \( x \in N \) it holds that \( D(x) = T_x N \). A sub-manifold \( N \) of \( M \) is called the maximal integral sub-manifold of \( D \) if \( N \) is connected, it is an integral sub-manifold of \( D \) and for each \( N' \) connected integral sub-manifold of \( D \) it holds that \( (N' \cap N \neq \emptyset \implies N' \subseteq N \) and \( N' \) is open in \( N \). If \( N \) is a maximal integral sub-manifold of \( D \) and \( x \in N \) then \( N \) is said to be the maximal integral sub-manifold of \( D \) passing through \( x \). If for each \( x \in M \) there exists a maximal integral sub-manifold of \( D \) passing through \( x \) then \( D \) is said to have the maximal integral sub-manifold property. There exists at most one maximal sub-manifold of \( D \) passing through \( x \in M \).

Let \( \mathcal{F} = \{ X_\gamma \mid \gamma \in \Gamma \} \) be a family of vector fields. The orbit of \( \mathcal{F} \) through a point \( x \in M \) is the set

\[
M^F_x = \{ X_1^{t_1} \circ X_2^{t_2} \circ \cdots \circ X_k^{t_k}(x) \mid X_i \in \mathcal{F}, t_i \in \mathbb{R}, i = 1, \ldots, k \}
\]

Let \( \mathcal{F} \) be a family of vector fields over \( M \). Define the distribution \( D_\mathcal{F} \) as

\[
D_\mathcal{F}(x) = \text{span}\{ X(x) \mid X \in \mathcal{F} \}.
\]

The distribution \( D \) is called \( \mathcal{F} \)-invariant if

\[
\begin{align*}
(1) & \quad \forall x \in M : D_\mathcal{F}(x) \subseteq D(x) \\
(2) & \quad \forall v \in D(x), \forall g : M \rightarrow M
\end{align*}
\]

\[
g(x) = X_1^{t_1} \circ X_2^{t_2} \circ \cdots \circ X_k^{t_k}(x) \implies \frac{dg}{dx}(x)v \in D(g(x))
\]

where \( X_i \in \mathcal{F}, t_i \in \mathbb{R}, i = 1, \ldots, k \)

Denote by \( P_\mathcal{F} \) the smallest \( \mathcal{F} \)-invariant distribution containing \( D_\mathcal{F} \). The main result of [4] is the following.

**Theorem 7 (Existence of maximal integral manifold)** For each \( x \in M \) the set \( M^F_x \) with a suitable topology and differentiable structure is a maximal integral sub-manifold of \( P_\mathcal{F} \). \( D_\mathcal{F} \) has maximal integral sub-manifold property if and only if \( D_\mathcal{F} = P_\mathcal{F} \).
Everything stated above also holds for analytic manifolds. For analytic manifolds the following, stronger result holds.

**Proposition 8** Let $M$ be an analytic manifold, let $\mathcal{F}$ be a family of analytic vector fields. Denote the smallest involutive distribution containing $D\mathcal{F}$ by $D^*_\mathcal{F}$. Then $D^*_\mathcal{F}$ has the maximal integral sub-manifold property. The maximal integral manifold of the distribution $D^*_\mathcal{F}$ passing through a point $x$ is the orbit of $\mathcal{F}$ passing through $x$, i.e $M^x_x$.

Let $M$ be a manifold, and let $\mathcal{F}$ be a family of vector fields over $M$. Let $x$ be an element of $M$. The reachable set of $\mathcal{F}$ from $x$ is defined as

$$ \text{Reach}(\mathcal{F}, x) = \{X_1^{t_1} \circ X_2^{t_2} \circ \cdots \circ X_k^{t_k}(x) | X_i \in \mathcal{F}, t_i \geq 0, i = 1, \ldots, k \} $$

Below the main results of [5] will briefly be recalled. Let $(G, \cdot)$ be a group, $p : G \to \mathbb{R}^n$ a function. Let $\cdot : G \times \mathbb{R} \to G$ be a surjective mapping. The triple $(G, p, \mathbb{R}^n)$ is called an abstract system. Let $a = (a_1, a_2, \ldots, a_p) \in G^p$, $\bar{b} = (b_1, b_2, \ldots, b_k) \in G^k$ and define $\psi_{\bar{a}}^k : \mathbb{R}^p \to \mathbb{R}^{kn}$ by

$$ \psi_{\bar{a}}^k(t) := \left[ p((t_1 \cdot a_1)(t_2 \cdot a_2) \cdots (t_p \cdot a_p)b_1), \ldots, p((t_1 \cdot a_1)(t_2 \cdot a_2) \cdots (t_p \cdot a_p)b_k) \right] $$

The abstract system $\Gamma$ is called smooth if $\psi_{\bar{a}}^k$ is a smooth map for all $\bar{a} \in G^p, \bar{b} \in G^k$. Denote by $D\psi_{\bar{a}}^k(t)$ the Jacobian of $\psi_{\bar{a}}^k$ at $t \in \mathbb{R}^p$. Then the rank of $p$ is defined to be $n = \sup_{\bar{a}, \bar{b}} \text{rank} D\psi_{\bar{a}}^k(t)$. A smooth representation of $\Gamma$ is a tuple $\Theta = (M, \{\phi_a | a \in G\}, h, x_0)$ where $M$ is a smooth Hausdorff manifold, not necessarily second-countable, $\phi_a : M \to M$ are diffeomorphisms for which $\phi_{ab} = \phi_b \circ \phi_a$ and $\phi_1 = \text{id}_M$ holds, $h : M \to \mathbb{R}^n$ is a smooth map, $x_0 \in M$ is the initial state. Further, for all $a = (a_1, a_2, \ldots, a_p) \in G^p$ define $\psi_a : \mathbb{R}^p \to M$ by $\psi_a(t) = \phi_{(t_1 \cdot a_1)(t_2 \cdot a_2) \cdots (t_p \cdot a_p)}(x_0)$. We require that $\psi_a$ to be smooth for all $a \in G^p$ and that $p(a) = h(\psi_a(x_0))$. If $\Theta = (M, \{\phi_a | a \in G\}, h, x_0)$ is a representation of the abstract system $\Gamma$, then $\psi_{\bar{a}}^k = \left[ h \circ \phi_{a_1} \circ \psi_{\bar{a}_2}, \ldots, h \circ \phi_{a_p} \circ \psi_{\bar{a}_k} \right]$.

A representation is called reachable if for $M = \{\psi_a(x_0) | a \in G\}$ holds. A representation is called transitive, if $\forall x, y \in M(\exists g \in G : y = \phi_g(x))$ holds. If $x = \phi_{g_1}(x_0)$ and $y = \phi_{g_2}(x_0)$ then $y = \phi_{g_1^{-1} g_2}(x)$. It means that a representation is transitive if and only if it is reachable. A representation is called distinguishable if for all $x_1 \neq x_2 \in M$ it holds that $h(\phi_{a}(x_1)) \neq h(\phi_{a}(x_2))$ for all $a \in G$. A transitive and distinguishable representation is called minimal. Let $\Theta_1 = (M_1, \{\phi_a^1 | a \in G\}, h^1, x^1_0)$ and $\Theta_2 = (M_2, \{\phi_a^2 | a \in G\}, h^2, x^2_0)$ be two smooth representations. A smooth map $\chi : M_1 \to M_2$ is a homomorphism from the representation $\Theta_1$ to the representation $\Theta_2$ if the following conditions hold: $\chi(x^1_0) = x^2_0, h^2 \circ \chi = h^1$ and $\phi^2_a \circ \chi = \chi \circ \phi^1_a$. In [5] the following theorem is proved.

**Theorem 9** Every smooth abstract system $(G, p, \mathbb{R}^n)$ with finite rank has a
minimal smooth representation \( \Theta = (M, \{ \phi_a \mid a \in G \}, h, x_0) \) with \( \dim M = \text{rank } p \). If \( \Theta' \) is a minimal smooth representation of \((G, p, \mathbb{R}^n)\), then there exists a homomorphism \( \chi \) \footnote{In \cite{5} \( \chi \) is claimed to be a diffeomorphism. However, the author of the current paper failed to see how this stronger statement follows from the proof presented in \cite{5}, unless \( M \) is second-countable.} \( \Theta \) to \( \Theta' \) such that \( \chi \) is a bijective map and \( \text{rank } \chi = \text{rank } p \).

4 Structure of the reachable set

Below we are going to apply the results from the previous section to determine the structure of the reachable set. The outline of the procedure is the following

- Given a linear switched system \( \Sigma \), we associate a family of vector fields \( \mathcal{F} \) over \( \mathbb{R}^n \) with it.
- Determine the smallest distribution \( D = P_\mathcal{F} \) invariant w.r.t. the family of vector fields constructed above. Find another family of vector fields \( \mathcal{F}' \) which spans the distribution.
- Consider the orbit \( M_0^\mathcal{F} \) of \( \mathcal{F} \) passing through 0. By Theorem 7 it is the maximal integral sub-manifold of \( P_\mathcal{F} \). But again by Theorem 7 and by uniqueness of maximal integral sub-manifold \( M_0^\mathcal{F} = M_0^{\mathcal{F}'} \).
- By direct computation we find the structure of \( M_0^{\mathcal{F}'} \) which turns out to be a subspace of \( \mathbb{R}^n \) in the case of linear switched systems. Moreover, computation shows that \( M_0^{\mathcal{F}'} = D(0) \). Therefore, by taking \( M_0^{\mathcal{F}'} \) with subspace topology, and proper differentiable structure, it will be a regular sub-manifold of \( \mathbb{R}^n \) and for each \( x \in M_0^{\mathcal{F}'} \) it holds that \( D(x) = T_x M_0^{\mathcal{F}'} \). Moreover, \( \dim M_0^{\mathcal{F}'} = \dim D(0) \).
- Consider the restriction \( \Sigma' \) of our switched system \( \Sigma \) to \( M_0^{\mathcal{F}'} \). Clearly, \[ \text{Reach}(\Sigma) = \text{Reach}(\Sigma') \subseteq M_0^{\mathcal{F}'} \]. Using the structure of \( M_0^{\mathcal{F}'} = M_0^{\mathcal{F}'} \), Theorem 6 can be proved, either by using the results of \cite{5} or by applying an elementary construction.

The rest of the subsection is devoted to carrying out the steps described above in a more formal way.

Consider a linear switched system \( \Sigma \). Assume that for each \( q \in Q \) and \( u \in \mathcal{U} \) the dynamics is given by \( \dot{x} = f_q(x, u) = A_q x + B_q u \). The family of vector fields \( \mathcal{F} \) associated with \( \Sigma \) is defined as

\[ \mathcal{F} = \{ A_q x + B_q u \mid q \in Q, u \in \mathcal{U} \} \]

The proof of the lemma below is given in the appendix.
Lemma 10 Consider a linear switched system \( \Sigma \) and the associated family of vector fields \( \mathcal{F} \). The smallest involutive distribution containing \( \mathcal{F} \) is of the following form

\[
D^*_{\mathcal{F}}(x) = \text{Span}\{A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_z u \mid i_1, i_2, \ldots i_k, z \in Q, j_1, j_2, \ldots j_k \geq 0, u \in \mathcal{U}\} \\
\cup \{[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots ]]x \mid i_1, i_2, \ldots i_k \in Q\}
\]

(2)

Lemma 11 Consider a linear switched system \( \Sigma \) and the family of associated vector fields \( \mathcal{F} \).

(a) The distribution \( D^*_{\mathcal{F}} \) has the maximal integral manifold property. The maximal integral manifold of \( D^*_{\mathcal{F}} \) passing through 0 is \( M^E_0 \).

(b) \( M^E_0 \) is of the form

\[
W := \text{Span}\{A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_z u \mid i_1, \ldots, i_k, z \in Q, j_1, \ldots, j_k \geq 0, u \in \mathcal{U}\}
\]

(3)

Proof Part (a)
Notice that \( \mathbb{R}^n \) is an analytic vector field. Besides, each member of \( \mathcal{F} \) is an analytic vector field. By Proposition 8 \( D^*_{\mathcal{F}} \) has the integral manifold property and its maximal integral manifold passing through 0 is equal to \( M^E_0 \). An alternative way to prove part (a) is to show that \( D^*_{\mathcal{F}} = W \) is \( \mathcal{F} \)-invariant.

Part (b) Consider the following family of vector fields:

\[
\mathcal{F} = \{A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_z u \mid i_1, \ldots i_k, z \in Q, u \in \mathcal{U}, j_1, \ldots j_k \geq 0\} \\
\cup \{[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots ]]x \mid i_1, \ldots i_k \in Q\}
\]

Then for all \( x \in \mathbb{R}^n \), \( D^*_{\mathcal{F}}(x) = \text{Span}\{X(x) \mid X \in \mathcal{F}\} = D^*_{\mathcal{F}}(x) \). Since \( D^*_{\mathcal{F}} \) has the maximal integral manifold property, part (ii) of Theorem 7 implies that \( P_{\mathcal{F}} = D^*_{\mathcal{F}} \). By part (i) of Theorem 7 the maximal integral manifold of \( D^*_{\mathcal{F}} = P_{\mathcal{F}} \) passing through 0 is the orbit of \( \mathcal{F} \) passing through 0 i.e. \( M^E_0 \). But by the part (a) of this lemma we get that the maximal integral manifold of \( D^*_{\mathcal{F}} \) passing through 0 is \( M^E_0 \). So we get that \( M^E_0 = M^E_0 \).

On the other hand, we shall show that \( M^E_0 \) indeed has the structure given by (3).

Assume \( X = A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_q u \). Then \( X'(z) = z + tA_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_q u \). So, if we identify each element of \( X \in W \) with a constant vector field, then we get that \( X'(0) = X, \mathcal{F} = W \cup \{[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots ]]x \mid i_1, \ldots i_k \in Q\} \) and \( W = \{X(0) \mid X \in W\} \subseteq M^E_0 \). We need to prove that \( M^E_0 \subseteq W \). Since
0 \in M_0^\mathcal{F} \cap W and

\[ M_0^\mathcal{F} = \{X_1^{t_1} \circ X_2^{t_2} \circ \cdots X_k^{t_k}(0) \mid X_i \in \mathcal{F}, t_i \in \mathbb{R}, i = 1, \ldots, k\} \]

it is sufficient to prove that \( W \) is invariant under \( \mathcal{F} \), i.e.

\[ \forall X \in \mathcal{F}, \forall t \in \mathbb{R}, \forall z \in W : X^t(z) \in W \]

If \( X = A_{i_1}^{j_1} A_{i_2}^{j_2} \cdots A_{i_k}^{j_k} B_q u \) then \( X^t(z) = z + tX(0) \in W \). Assume that \( X = [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] x \). Assume that \( z \in W \). By definition of \( X^t \) and Cayley-Hamilton theorem we get

\[ X^t(z) = \exp([A_{i_1}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] t)z \]

\[ = \sum_{j=0}^{n-1} g_j(t) [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] ^j z \]

It is easy to see that \( [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] \in \text{Span}\{A_{i_1} A_{i_2} \cdots A_{i_k} \mid z_1, \ldots, z_k \in Q\} \), which implies

\[ z \in W \implies [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}] \cdots] z \in W \]

Then it follows easily that \( z \in W \implies X^t(z) \in W \).

**Proof of Theorem 6** It is sufficient to prove that \( \text{Reach}_{\text{const}}(\Sigma) = W \). Indeed, since \( W \) is a subspace of \( \mathbb{R}^n \), it is closed in \( \mathbb{R}^n \), so, in this case we get \( W = Cl(\mathcal{F}w) = Cl(\text{Reach}_{\text{const}}(\Sigma)) = \text{Reach}(\Sigma) \). Let \( \mathcal{F} \) be the family of vector fields associated to \( \Sigma \). For \( X_i = A_{q_i} x + B_{q_i} u_i \in \mathcal{F}, t_i \in \mathbb{R}, i = 1, 2, \ldots, k, k \geq 0 \) denote

\[ X_1^{t_1} \circ X_2^{t_2} \circ \cdots \circ X_k^{t_k}(x_0) = x(x_0, u_1 u_2 \cdots u_k, q_1 q_2 \cdots q_k, t_1 t_2 \cdots t_k) \]

It follows that \( \text{Reach}(\mathcal{F}, 0) = \text{Reach}_{\text{const}}(\Sigma) \). On the other hand \( \text{Reach}(\mathcal{F}, 0) \subseteq M_0^\mathcal{F} \). From Lemma 11 we get that \( M_0^\mathcal{F} = W \). Let \( n = \dim W \) and let \( b_1, \ldots, b_n \) be a basis of \( W \). Let \( T : W \to \mathbb{R}^n \) be a linear isomorphism. It follows that for each \( b_i, i = 1, \ldots, n \) there exists vector fields \( X_{i,1}, \ldots X_{i,n_i} \in \mathcal{F}, n_i \geq 0 \) such that \( b_i = X_{i,n_i}^{t_{i,n_i}} \circ X_{i,n_i-1}^{t_{i,n_i-1}} \cdots X_{i,1}^{t_{i,1}}(0) \) for some \( t_{i,1}, \ldots, t_{i,n_i} \in \mathbb{R} \). Assume that \( X_{i,j} = A_{q_{i,j}} x + B_{q_{i,j}} u_{i,j} \). Define \( u_i = u_{i,1} \cdots u_{i,n_i}, w_i = q_{i,1} \cdots q_{i,n_i} \), \( \tau_i = \tau_{i,1} \cdots \tau_{i,n_i} \). With the notation above we get that \( x(0, u_i, w_i, \tau_i) = b_i \). For any sequence \( s = s_1 \cdots s_k \) let \( s = s_k s_{k-1} \cdots s_1 \), and \( -s = (-s_1)(-s_2) \cdots (-s_k) \). Then define the sequences \( w = \bar{w}_1 w_1 \cdots \bar{w}_{n-1} w_{n-1} \bar{w}_n \), \( \tau = (\bar{\tau}_1 \tau_1(\bar{\tau}_2 \tau_2(\cdots (\bar{\tau}_{n-1} \tau_{n-1}(\bar{\tau}_n \tau_n) \cdots s_2)(-s_1) \cdots s_1) \cdots s_1) \cdots s_1) \cdots s_1) \cdots s_1 \). Let \( v_i = O_i(O_1 \cdots O_{i-1}O_{i-1}O_i u_i O_{i+1}O_{i+1} \cdots O_n) \), where \( O_i = 0 \cdots 0 \in \mathcal{U}^{(w_i)}, i = 1, \ldots, n \). Then it is easy to see that

\[ x(0, v_i, w, \tau) = b_i, \quad i = 1, \ldots, n \]
Indeed, \( x(0, v_i, w, \tau) = x(y_i, s_i, \beta_i', \gamma_i') \), where \( y_i = x(x(0, s_i, \beta_i, \gamma_i), u_i, w_i, \tau_i) \), 
\( s_i = \bigodot_i \bigodot_{i-1} \bigodot_{i-2} \cdots \bigodot_1 0 \), 
\( \gamma_i = (\overline{0} \tau_i \cdots (\overline{0} t_i) \tau_i (\overline{0} \tau_i) \beta_i = \overline{w}_i \), 
\( w_1 \cdots \overline{w}_{i-1} w_i \cdots \overline{w}_{n-1} w_n \). It is easy to see that for any \((s, d) \in (Q \times \mathbb{R})^*\), 
\( x(0, \bigodot_{[s]} s, v) = 0 \), \( \bigodot_{[s]} = 0 \cdots 0 \in \mathcal{U}^{|s|} \). Thus, \( x(0, s_i, v_i, \gamma_i) = 0 \), and \( y_i = x(0, u_i, w_i, \tau_i) = b_i \). It is easy to see that for all \((u, s, d) \in (\mathcal{U} \times \mathbb{Q} \times \mathbb{R})^*\), 
\( x(y, u, s, (\overline{d}) d) = y, y \in W \). That is, by noticing that \( \bigodot_i = \bigodot_i \), we get that \( x(y, s'_i, \beta_i', \gamma_i') = y, y \in W \), thus \( x(0, v_i, w, \tau) = x(b_i, s'_i, \beta_i', \gamma_i') = b_i \). Let \( N = 2n \) and define the function \( M : \mathbb{R}^N \to \mathbb{R}^{n \times n} \) by 
\[
M(\eta) = \left[ T x(0, v_1, w, \eta), \ldots, T x(0, v_n, w, \eta) \right]
\]
Then \( \eta \mapsto \det M(\eta) \) is an analytic functions and \( \det M(\tau) \neq 0 \). By the well-known properties of analytic functions there exists a \( \psi = (\psi_1, \ldots, \psi_N) \in \mathbb{R}^N \), \( \psi_1, \ldots, \psi_N \geq 0 \) such that \( \det M(\psi) \neq 0 \), that is, \( \text{rank} M(\psi) = n \). It implies that 
\[ W = T^{-1}(\mathbb{R}^n) = \text{Span}\{ x(0, v_i, w, \psi) \mid i = 1, \ldots, n \} = \left\{ x(0, \sum_{i=1}^n a_i v_i, w, \psi) \mid a_1, \ldots, a_n \in \mathbb{R} \right\} \subseteq \text{Reach}(\Sigma) \], therefore 
\[ \left\{ x(0, u, w, \psi) \mid u \in \text{PC}_{\text{const}}(T, \mathcal{U}) \right\} = W = \text{Reach}(\Sigma) \]
That is, we get part (b) of the theorem, which implies part (a).

An alternative approach will be presented below. This approach uses the results from [5]. We proceed by proving part (b) of theorem, which already implies part (a). Define \( G = (\mathcal{U} \times \mathbb{Q} \times \mathbb{R})^*/\sim \), where \( \sim \) is the smallest congruence relation such that \((u, q, 0) \sim 1 \) and \((u, q, t_1)(u, q, t_2) \sim (u, q, t_1 + t_2) \). Denote by 
\[ [(u, w, \tau)] \in G \] 
The equivalence class represented by \((u, w, \tau) \in (\mathcal{U} \times \mathbb{Q} \times \mathbb{R})^* \). The definition of \( G \) is essentially identical to the definition of the group of piecewise-constant inputs in [5]. Define the map \( Z : \mathcal{X} \times (\mathcal{U} \times \mathcal{Q} \times \mathcal{T})^* \to \mathcal{X} \) by 
\[ Z(x, u, w, \tau) := x(x, u, w, \tau) \]. It is clear that the dependence of \( Z \) on the switching times is analytic, i.e. \( \forall x \in U^*, w \in Q^* \), \( \in \mathcal{X} \) : 
\[ Z(x, u, w, \tau) : T^{[w]} \to \mathcal{X} \] is analytic . From Proposition 5 it is clear that by the principle of analytic continuation \( Z(x, u, w, \tau) \) can be extended to \( \mathbb{R}^{[w]} \). From now on we will identify \( Z \) with this extension. Then it is easy to see that \( Z \) is in fact a function on \( G \), since \((u, w, \tau) \sim (u', w', \tau') \) \( \implies \) \( Z(x, u, w, \tau) = Z(x, u', w', \tau') \) for all \( x \in \mathcal{X} \). Define 
\[ \Theta = (W, \{ \phi \mid A \in G \}, 0, id) \]
where \( W = M_0^F \) as above and \( \phi_{[(a, u, w), \tau]}(x) = Z(x, u, w, \tau) \). Now, define \( \cdot : G \times \mathbb{R} \to G \) by \([[(u, w, \tau)] \cdot \alpha = [a(u, w, \tau)] \]. It is easy to see that \( \Theta \) is a smooth representation of \( R \) with respect to \( \cdot \), \( \Theta \) is transitive and distinguishable, thus minimal. Recall the definition of the function \( \psi_{\mu}^R \) from Section 3. Let 
\[ d = \text{rank} R = \text{sup}_{x \in L} \text{rank} \psi_{\mu}^R(x) \]. We want to show that \( d = \text{dim} W = n \). Let \( \Theta_m = (M_m, \{ \phi_{\alpha}^m \mid a \in G \}, h^m, x_0^m) \) be a minimal smooth representation of \( R \) w.r.t \( \cdot \), such that \( \text{dim} M_m = d \) as described in Theorem 9. Let \( \chi : \)
$M_m \to W$ the representation homomorphism described in Theorem 9. We shall prove that $\chi$ is a diffeomorphism. Since $W$ is a second-countable Hausdorff manifold, we get that $W$ has a positive-definite Riemannian structure. Since $\chi$ is analytic, Proposition 9.4.2 of [8] implies that $M_m$ has a positive-definite Riemannian structure. We shall show that $M_m$ is connected. If $M_m$ is connected and has a positive-definite Riemannian structure, then $M_m$ is a second countable Hausdorff manifold by Proposition 10.6.4 of [8]. But then bijectivity of $\chi$ implies that $\dim M_m = \dim W = d = n$. To see that $M_m$ is connected, notice that for any $g = [(a, w, \tau)] \in G$ it holds that $R((0 \cdot g)[(s, v, t)]) = x(0, 0s, wv, \tau t) = R([(s, v, t)])$. That is, $h^m \circ \phi^m_{[s,v,t]} \circ \psi^m_g(0) = R([(s, v, t)]) = h^m \circ \phi^m_{[s,v,t]}(x_0)$. Since $\Theta_m$ is indistinguishable, it implies that $\theta^m_g(0) = x_0$. For any $x \in M_m$ there exists a $g'$ such that $\theta^m_{g'}(x_0) = x$, by transitivity of $\Theta_m$. But then there exists $g, \alpha$ such that $\alpha \cdot g = g'$. Since $\Theta_m$ is a smooth representation, the map $\theta^m_g$ is smooth, therefore continuous, which implies that $\theta^m_{g'}(\mathbb{R})$ is connected. That is, $x_0 = \theta^m_g(0)$ and $x = \theta^m_{g'}(\alpha)$ are in the same connected component of $M_m$. Since $x$ is an arbitrary element of $M_m$, we get that $M_m$ is connected.

Now, let $a = (a_1, a_2, \ldots, a_k) \in G^k, b = (b_1, b_2, \ldots, b_p) \in G^p, \mu \in \mathbb{R}^k$ such that $\operatorname{rank} \operatorname{D} \theta^m_{\mu}(\mu) = n$. Assume that $a_j = [(s_j, r_j, \gamma_j)] \in G$ and $b_i = [(v_i, w_i, \sigma_i)] \in G$. For all $z = z_1 z_2 \cdots z_k \in Q^*$ and $\tau = \tau_1 \tau_2 \cdots \tau_k$ denote by $\operatorname{exp}(A_z \tau)$ the expression $\operatorname{exp}(A_{z_1} \tau_1) \operatorname{exp}(A_{z_2} \tau_2 - 1) \cdots \operatorname{exp}(A_{z_k} \tau_k)$. For each $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$, let $M_j(t) = x(0, s_j 00 \cdots 0, r_j r_{j+1} \cdots r_k, t_j t_{j+1} \cdots t_k)$. We get that

$$D \theta^m_{\mu}(\mu) = D_{\mu_1, \mu_2, \ldots, \mu_k} \theta_{(a_1, \mu_1)(a_2, \mu_2) \cdots (a_k, \mu_k)}(0) = D_{\mu_1, \mu_2, \ldots, \mu_k} x(0, v_i, w_i, \sigma_i) +$$

$$+ \operatorname{exp}(A_{w_i} \sigma_i) x(0, (\mu_1 s_1)(\mu_2 s_2) \cdots (\mu_k s_k), r_1 \cdots r_k, \gamma_1 \cdots \gamma_k) =$$

$$= D_{\mu_1, \mu_2, \ldots, \mu_k} \operatorname{exp}(A_{w_i} \sigma_i) \sum_{j=1}^{k} \mu_j \cdot z(0, s_j 00 \cdots 0, r_j r_{j+1} \cdots r_k, \gamma_j \gamma_{j+1} \cdots \gamma_k)$$

$$= \operatorname{exp}(A_{w_i} \sigma_i) M(\gamma)$$

where $\gamma = (\gamma_1, \ldots, \gamma_k)$ and $M(\gamma) = [M_1(\gamma), M_2(\gamma), \ldots, M_k(\gamma)]$. Thus,

$$D \theta^m_{\mu}(\mu) = \begin{bmatrix} \exp(A_{w_1} \sigma_1) & 0 & \cdots & 0 \\ 0 & \exp(A_{w_2} \sigma_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp(A_{w_k} \sigma_k) \end{bmatrix} M(\gamma)$$

It follows that $n = \operatorname{rank} D \theta^m_{\mu}(\mu) = \operatorname{rank} M(\gamma)$. Notice that the dependence of $M(t)$ on $t$ is analytic. Then it follows that we can choose $t \in T^k$ such that
rank $M(t) = n$. Since $\sum_{j=1}^{k} \alpha_j M_j(t) = x(0, (\alpha_1 s_1) \cdots (\alpha_k s_k), r_1 \cdots r_k, t_1 \cdots t_k)$ and $\dim \text{Im} M = \dim \text{Reach}(\Sigma)$, it follows that

$$\text{Reach}(\Sigma) = \text{Im} M = \{x(0, u(.), (r_1, t_1)(r_2, t_2) \cdots (r_k, t_k)) \mid u(.) \in PC(T, U)\}$$

5 Conclusions

The structure of the reachable set for linear switched systems has been derived in the paper. The derivation relies on techniques from differential geometric theory of nonlinear systems. The author would like to investigate the application of nonlinear techniques to more general classes of hybrid systems. The hope is that a geometric theory may emerge for some classes of hybrid systems. As a first step toward such a theory see [9].

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A Appendix

Proof of Lemma 10 The following two facts will be used in the proof.

- Let $X, Y$ be vector fields over $\mathbb{R}^n$ of the form $X(x) = Ax$, $Y(x) = y$ for some $A \in \mathbb{R}^{n \times n}$ and $y \in \mathbb{R}^n$. Then in the usual coordinates $[X, Y](x) = -Ay$.
- For $i = 1, 2, \ldots, k$ let $X_i$ be vector fields of the form $X_i(x) = A_i x$. Then

$$[X_1, [X_2, \cdots [X_{k-1}, X_k]\cdots ]](x) \in \text{Span}\{A_{\pi(1)} A_{\pi(2)} \cdots A_{\pi(k)} \mid \pi(1), \pi(2), \ldots, \pi(k) \in \{1, 2, \ldots, k\}\}$$

Clearly, $D_x^x = \text{Span}\{[f_1, [f_2, \cdots [f_{k-1}, f_k]\cdots ] \mid f_i \in \mathcal{F}, i = 1, 2, \ldots, k\}$. Denote the right-hand side of (2) by $D$. First $D \subseteq D_x^x$ will be proved. Since $A_q x + B_q 0 = A_q x + F$ then we get that $[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}]\cdots ] x \in D_x^x$ for all $i_1, \ldots, i_k \in Q$. Clearly $[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k} \cdots x + B_{i_k} u_k \cdots ]](x)$ belongs to $D_x^x$. But by linearity of the Lie-brackets we get

$$[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k} x + B_{i_k} u_k \cdots ]](x) = [A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}]\cdots ](x) - A_{i_1} A_{i_2} \cdots A_{i_{k-1}} B_{i_k} u_k$$

From this and the fact that $[A_{i_1}, [A_{i_2}, \cdots [A_{i_{k-1}}, A_{i_k}]\cdots ] x \in D_x^x$ we get that $A_{i_1} A_{i_2} \cdots A_{i_{k-1}} B_{i_k} u_k \in D_x^x$ for all $i_1, \cdots, i_k \in Q$ and $u_k \in \mathcal{U}$. So we get that
$D \subseteq D \ast$. The reverse inclusion $D \ast \subseteq D$ will be shown by proving that for all $f_1, \cdots, f_k \in \mathcal{F}$ the vector field $[f_1, [f_2, \cdots [f_{k-1}, f_k] \cdots]$ belongs to $D$. This is done by induction on the length of expression. For $k = 1$ it is true, since $\mathcal{F} \subseteq D$. Assume it is true for all expression of length $\leq k$. Consider the expression $[f_1, [f_2, \cdots [f_k, f_{k+1}] \cdots]$. The vector field $[f_2, [f_3, \cdots [f_k, f_{k+1}] \cdots]$ belongs to $D$. By linearity of Lie-brackets it is enough to prove that for all $f = Aq x + Bq u$ and for all $Y = A_{i_1} A_{i_2} \cdots A_{i_l} Bz w$ or $Y = [A_{i_1}, [A_{i_2}, \cdots [A_{i_{l-1}}, A_{i_l}] \cdots]$ it holds that $[f, Y] \in D$. For the first case we get

$$[Aq x + Bq u, Y] = [Aq x, Y] + [Bq u, Y] = [Aq x, A_{i_1} A_{i_2} \cdots A_{i_l} Bz w] + [Bq u, A_{i_1} A_{i_2} \cdots A_{i_l} Bz w] = -Aq A_{i_1} A_{i_2} \cdots A_{i_l} Bq w$$

For the second case we get that

$$[Aq x + Bq u, Y] = [Aq x, Y] + [Bq u, Y] = [Aq x, [A_{i_1}, [A_{i_2}, \cdots [A_{i_{l-1}}, A_{i_l}] \cdots] x] + [Bq u, [A_{i_1}, [A_{i_2}, \cdots [A_{i_{l-1}}, A_{i_l}] \cdots] x] = [Aq x, [A_{i_1}, [A_{i_2}, \cdots [A_{i_{l-1}}, A_{i_l}] \cdots] x] + [A_{i_1}, [A_{i_2}, \cdots [A_{i_{l-1}}, A_{i_l}] \cdots] Bq u \in D$$

References


