Isotropic Multiresolution Analysis: Theory and Applications

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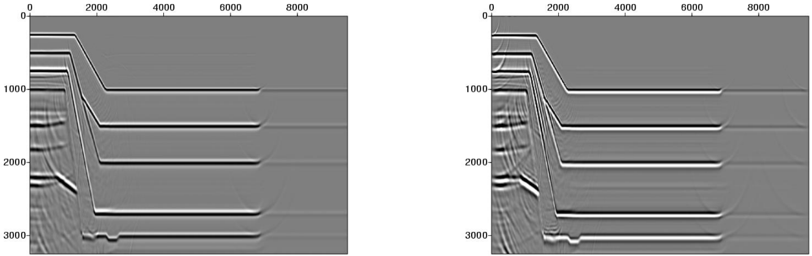
Collaborators & Acknowledgements

This work has been performed in collaboration with Manos Papadakis (UH), Juan R. Romero (U.Puerto Rico), Simon K. Alexander (UH), Shikha Baid (formerly UH).

We gratefully acknowledge the contributions and previous related work by Robert Azencott (UH and ENS) and Bernhard G. Bodmann (UH).

Outline

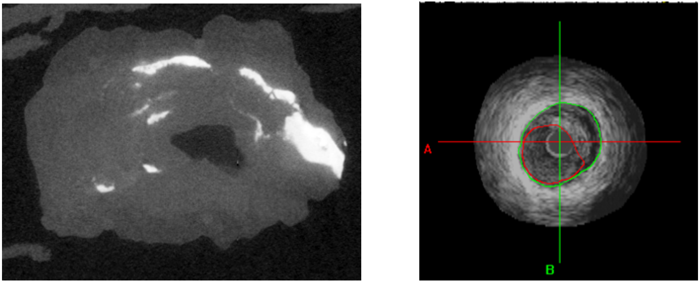
Example from Seismic Imaging



(a) (b)

Figure: Vertical cross-sections of volumes produced using radial (left) and non-radial (right) filters. (Courtesy: Total E&P, USA)

Example from Biomedical Imaging

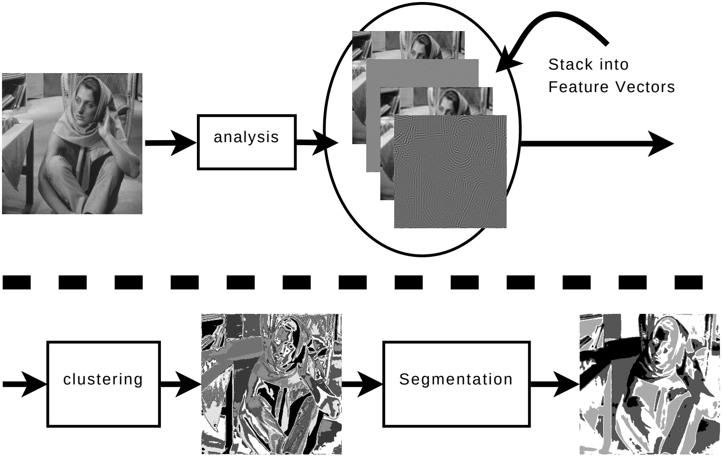


(a) 2D slice from 3D μCT x-ray data

(b) Slice from Intravascular Ultra Sound data

Figure: Examples of medical 3D data sets.

Segmentation flowchart



Problems with tensor product basis

'Barbara'

DB8 decomposition

Problems with tensor product basis

'Barbara'

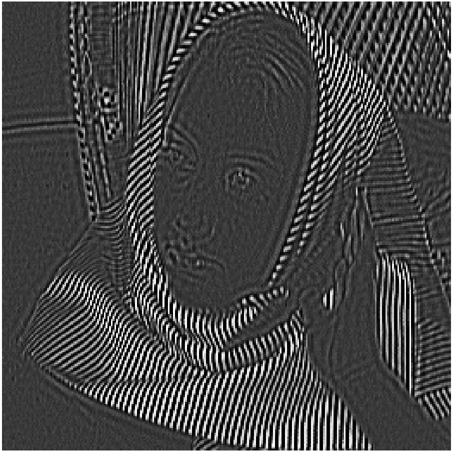
DB8 high pass

Problems with tensor product basis

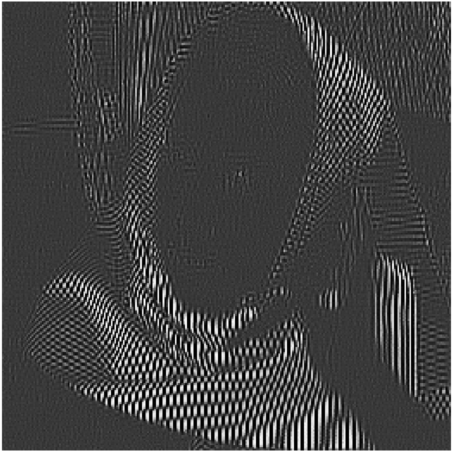
IMRA high pass

DB8 high pass

Problems with tensor product basis (Zoom)



IMRA high pass



DB8 high pass

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Notation and Preliminaries

- We say that $\Omega \subset \mathbb{R}^d$ is radial if for all $O \in SO(d)$, $O\Omega = \Omega$.

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- A unitary dilation operator with respect to $\mathfrak{M}$ is defined by $D_{\mathfrak{M}} f(x) = |\det \mathfrak{M}|^{1/2} f(\mathfrak{M}x)$.
- For $y \in \mathbb{R}^d$, the (unitary) shift operator T_y is defined by $T_y f(x) = f(x - y)$.
- A function $f \in L^2(\mathbb{R}^d)$ is said to be isotropic if there exists a $y \in \mathbb{R}^d$ and a radial function $g \in L^2(\mathbb{R}^d)$ such that $f = T_y g$.

Definition

An IMRA is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ satisfying the following conditions:

- $\forall j \in \mathbb{Z},\ V_j \subset V_{j+1}$,
- $(D_{\mathfrak{M}})^j V_0 = V_j$,
- $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$,
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If P_0 is the orthogonal projection onto V_0, then

$$\mathcal{O}P_0 = P_0\mathcal{O} \qquad \text{for all } O \in SO(d),$$

where $\mathcal{O}$ is the unitary operator given by $\mathcal{O}f(x) := f(O^T x)$

Characterization of V_0

Theorem

Let V be an invariant subspace of $L^2(\mathbb{R}^d)$ under the action of the translation group induced by $\mathbb{Z}^d$. Then V remains invariant under all rotations if and only if $V = PW_\Omega$ for some radial measurable subset Ω of $\mathbb{R}^d$.

Characterization of IMRAs

Theorem

Let $\mathfrak{M}$ be a radially expansive matrix and $C := \mathfrak{M}^*$. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ is an IMRA with respect to $\mathfrak{M}$ if and only if $V_j = PW_{C^j\Omega}$, where Ω is radial and satisfies

1. $\Omega \subset C\Omega$.
2. *The set-theoretic complement of* $\cup_{j=1}^{\infty} C^j\Omega$ *is null.*
3. $\lim_{j\to\infty} |C^{-j}\Omega| = 0$.

Moreover the only singly generated IMRAs are precisely $V_j = PW_{C^j\Omega}$, where Ω is a radial subset of $\mathbb{T}^d$ satisfying (1), (2) and (3).

Characterization of isotropic refinable functions

Definition

A function ϕ in $L^2(\mathbb{R}^d)$ is called refinable with respect to dilations induced by an expansive matrix $\mathfrak{M}$, if there exists an $H \in L^\infty(\mathbb{T}^d)$ such that

$$\widehat{\phi}(\mathfrak{M}^*\xi) = H(\xi)\widehat{\phi}(\xi), \quad \text{for a.e.} \quad \xi \in \mathbb{R}^d.$$

The function H is called the low-pass filter or mask corresponding to ϕ.

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Theorem

If $\phi \in L^2(\mathbb{R}^d)$ is a refinable function which is also isotropic and

$$\lim_{\xi \to 0} \widehat{\phi}(\xi) = L \neq 0,$$

then $\phi \in PW_{\rho/(\rho+1)}$, where ρ is the dilation factor of $\mathfrak{M}$.

Consequences

Corollary

There are no isotropic, refinable functions that are compactly supported in the spatial domain.

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Corollary

If the Fourier transform of ϕ *is supported outside the ball of radius* $1/2$ *centered at the origin then the MRA generated by* ϕ *is not an IMRA.*

Examples of IMRA

Example

The sequence of closed subspaces $V_j = PW_{2^j B(0,\rho)}$, *for any* $\rho > 0$ *and* $j \in \mathbb{Z}$ *is an IMRA.*

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Let $B(0,r,s)$ *denote the (2-dimensional) annulus centered at the origin having inner radius* r *and outer radius* s.

Define the sets $\mathbf{A} = \bigcup_{n=1}^{\infty} B(0, r_n, 2^{n-1})$, *with* $r_n = 2^{n-1} - (1/16)^n$, *and* $\mathbf{B} = B(0, 1/2)$.

Define $\Omega := \mathbf{A} \bigcup \mathbf{B}$.

Extension Principles

Let ϕ be a refinable function such that $\hat{\phi}$ is continuous at the origin and,

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Furthermore, let H_i, $i = 1, \ldots, m$, be $\mathbb{Z}^d$-periodic measurable functions and define m wavelets, ψ_i via

$$D^*_{\mathfrak{M}} \hat{\psi}_i = H_i \hat{\phi}.$$

Unitary Extension Principle

Theorem (Daubechies, Han, Ron and Shen 2003)

Assume $H_i \in L^\infty(\mathbb{T}^d)$ for all $i = 0, \ldots, m$, then the following two conditions are equivalent:

1. *The set $\left\{ D^j_{\mathfrak{M}} T_k \psi_i : j \in \mathbb{Z}, k \in \mathbb{Z}^d, i = 1, \ldots, m \right\}$ is a Parseval Frame for $L^2(\mathbb{R}^d)$.*
2. *For all $\xi \in \sigma(V_0)$,*
   - $\lim_{j \to -\infty} \Theta(\mathfrak{M}^{*j}\xi) = 1$.
   - *If $q \in (\mathfrak{M}^{*-1}\mathbb{Z}^d)/\mathbb{Z}^d \setminus \{0\}$ and $\xi + q \in \sigma(V_0)$, then*

$$\Theta(\mathfrak{M}^*\xi)H_0(\xi)\overline{H_0(\xi + q)} + \sum_{i=1}^{m} H_i(\xi)\overline{H_i(\xi + q)} = 0.$$

Fundamental function

Θ is the so-called fundamental function, defined by

$$\Theta(\xi) = \sum_{j=0}^{\infty} \sum_{i=1}^{m} \left| H_i(\mathfrak{M}^{*j}\xi) \right|^2 \prod_{l=0}^{j-1} \left| H_0(\mathfrak{M}^{*l}\xi) \right|^2 .$$

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Θ_M is the so-called mixed fundamental function, defined by

$$\Theta_M(\xi) = \sum_{j=0}^{\infty} \sum_{i=1}^{m} H_i^s(\mathfrak{M}^{*j}\xi) \overline{H_i^a(\mathfrak{M}^{*j}\xi)} \prod_{l=0}^{j-1} H_0^s(\mathfrak{M}^{*l}\xi) \overline{H_0^a(\mathfrak{M}^{*l}\xi)}.$$

Mixed Extension Principle

Theorem (Daubechies, Han, Ron and Shen, 2003)

Assume $H_i^a, H_i^s \in L^\infty(\mathbb{T}^d)$ *for all* $i = 0, \ldots, m$. *Then the following two conditions are equivalent,*

1. *The sets* $\Psi^a := \left\{ D^j_{\mathfrak{M}} T_k \psi_i^a : j \in \mathbb{Z}, k \in \mathbb{Z}^d, i = 1, \ldots, m \right\}$ *and* $\Psi^s := \left\{ D^j_{\mathfrak{M}} T_k \psi_i^s : j \in \mathbb{Z}, k \in \mathbb{Z}^d, i = 1, \ldots, m \right\}$ *is a pair of dual frames for* $L^2(\mathbb{R}^d)$.
2. *For all* $\xi \in \sigma(V_0^a) \bigcap \sigma(V_0^s)$,
   - Ψ^a *and* Ψ^s *are Bessel families.*
   - $\lim_{j \to -\infty} \Theta_M(\mathfrak{M}^{*j}\xi) = 1$.
   - *If* $q \in (\mathfrak{M}^{*-1}\mathbb{Z}^d)/\mathbb{Z}^d \setminus \{0\}$ *and* $\xi + q \in \sigma(V_0^a) \bigcap \sigma(V_0^s)$, *then*

$$\Theta_M(\mathfrak{M}^*\xi) H_0^s(\xi) \overline{H_0^a(\xi + q)} + \sum_{i=1}^{m} H_i^s(\xi) \overline{H_i^a(\xi + q)} = 0.$$

Our versions of Extension Principles

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$$X_{\phi\psi} := \left\{ D^j_{\mathfrak{M}} T_k \psi_i : j \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}^d, i = 1, \ldots, m \right\} \cup \left\{ T_k \phi : k \in \mathbb{Z}^d \right\}.$$

Our version of Unitary Extension Principle

Theorem

Assume $H_i \in L^\infty(\mathbb{T}^d)$ for all $i = 1, \ldots, m$ then $X_{\phi\psi}$ is a Parseval frame for $L^2(\mathbb{R}^d)$ if and only if for all $q \in (\mathfrak{M}^{*-1}\mathbb{Z}^d)/\mathbb{Z}^d$ and for a.e. $\xi, \xi + q \in \sigma(V_0)$,

$$\sum_{i=0}^{m} H_i(\xi)\overline{H_i(\xi + q)} = \delta_{q,0}.$$

Our version of Mixed Extension Principle

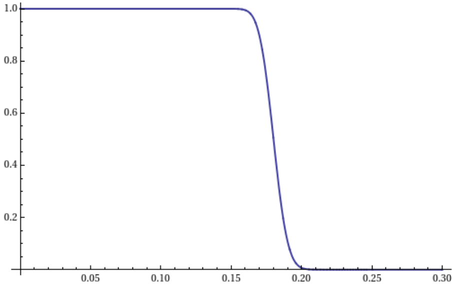
Theorem

Let H_i^a, $H_i^s \in L^\infty(\mathbb{T}^d)$ for $i = 1, \ldots, m$, then $X_{\phi\psi}^a$ and $X_{\phi\psi}^s$ form a pair of dual frames for $L^2(\mathbb{R}^d)$ if and only if

1. *$X_{\phi\psi}^a$ and $X_{\phi\psi}^s$ are Bessel families,*
2. *For all $q \in (\mathfrak{M}^{*-1}\mathbb{Z}^d)/\mathbb{Z}^d$ and for a.e. $\xi, \xi + q \in \sigma(V_0^a) \cap \sigma(V_0^s)$,*

$$\sum_{i=0}^{m} H_i^s(\xi)\overline{H_i^a(\xi + q)} = \delta_{q,0}.$$

An example of dual isotropic wavelet frames



- $H_0^a = 1$ inside the ball of radius b_2
- $H_0^a = 0$ on $\mathbb{T}^d \setminus B(0, b_1)$
- $H_0^a|_{B(0,b_1)}$ is radial.

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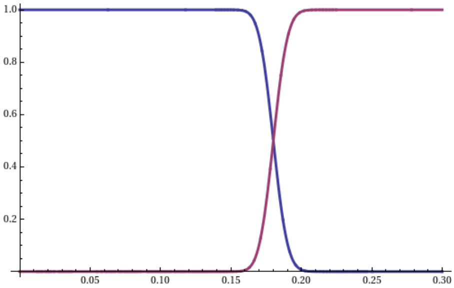

Let h^a be a $\mathbb{Z}^d$-periodic function defined by,

$$h^a(\xi) = \frac{1 - H_0^a(\xi)}{|\det(\mathfrak{M})|^{1/2}}.$$

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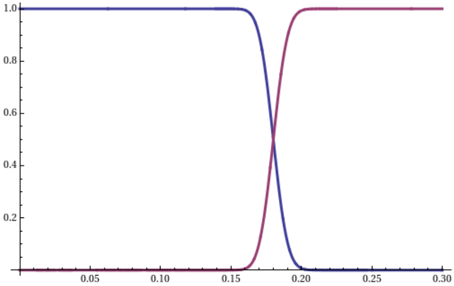

Using h^a we define the analysis high pass filters

$$H_i^a(\xi) := e^{-2\pi i \langle q_{i-1}, \xi \rangle} h^a(\xi).$$

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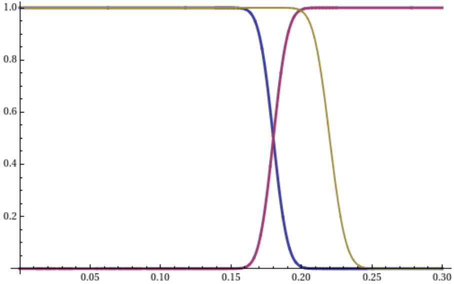

- $H_0^s = 1$ inside the ball of radius b_1
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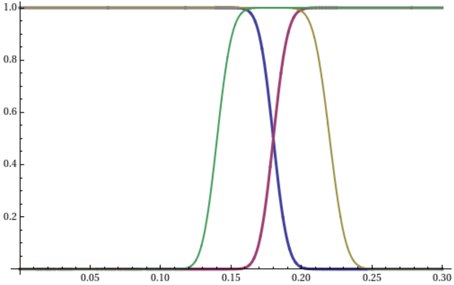

- $h^s = 0$ inside the ball of radius b_3, for some $b_3 > 0$
- $h^s = \frac{1}{|\det(\mathfrak{M})|^{1/2}}$ on $\mathbb{T}^d \setminus B(0, b_2)$
- $h^s|_{\mathbb{T}^d}$ is radial

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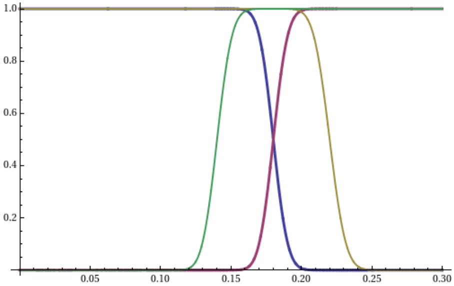
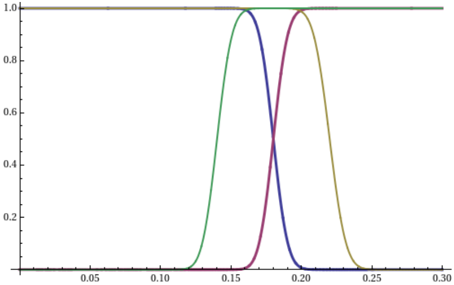


Using h^s we define the synthesis high pass filters:

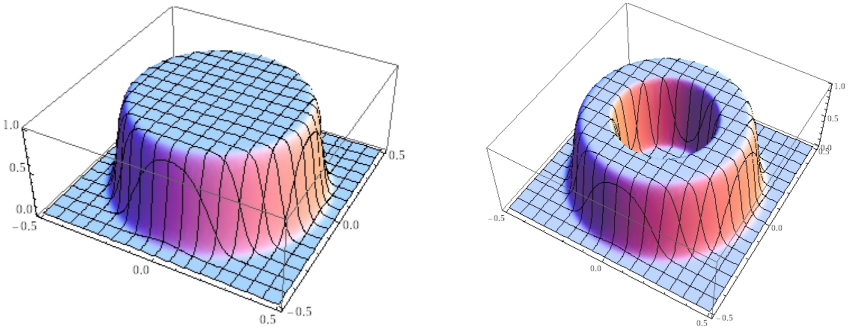
$$H_i^s(\xi) = e^{-2\pi i \langle q_{i-1}, \xi \rangle} h_s(\xi)$$

An example of dual isotropic wavelet frames



$$\sum_{i=0}^{m} H_i^s(\xi)\overline{H_i^a(\xi+q)} = \delta_{q,0}.$$

2D IMRA scaling function and wavelet



(a) Fourier transform of the scaling function $\hat{\phi}$

(b) Fourier transform of the wavelet $\hat{\psi}_1(2.)$

Outline

Feature spaces and feature maps

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- The vector space of all V-valued functions h defined on $\mathbb{R}^3$ such that $\|h(\cdot)\|$ is square integrable on $\mathbb{R}^3$ is denoted by H. Thus, $\|h\| = \left(\int_{\mathbb{R}^3} \|h(z)\|^2 dz\right)^{1/2}$.

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- The vector space of all V-valued functions h defined on $\mathbb{R}^3$ such that $\|h(\cdot)\|$ is square integrable on $\mathbb{R}^3$ is denoted by H. Thus, $\|h\| = \left(\int_{\mathbb{R}^3} \|h(z)\|^2 dz\right)^{1/2}$.
- Let $A : L^2(\mathbb{R}^3) \to H$ be a bounded linear transformation that associates to each f the V-valued function Af defined on $\mathbb{R}^3$. This linear transformation is called a feature map.

Steerability

Definition

Consider a bounded linear feature mapping A generating for each image f and each voxel $z \in \mathbb{R}^3$ a feature vector $Af(z)$ in the Euclidean space V. We shall say that A is a steerable feature mapping if there is a mapping U from the group G of rigid motions into the general linear group $GL(V)$ of V such that for each rigid motion R of $\mathbb{R}^3$, the invertible transformation $U(R)$ verifies,

$$A[\mathcal{R}f](z) = U(R)\left[Af\left(R(z)\right)\right] \quad z \in \mathbb{R}^3.$$

Here $\mathcal{R}$ is the following transformation induced by R on $L^2(\mathbb{R}^3)$:

$$\mathcal{R}f(z) = f(Rz).$$

An IMRA based steerable feature map

Using the functions ϕ and ψ, we define the following feature map:

$$Af(z) = (< f,\ T_z\phi(\cdot/8) >,\ \ < f,\ T_z\psi(\cdot/4) >,\ \ < f,\ T_z\psi(\cdot/2) >)\,.$$

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- This map is both translation and rotation invariant
- Hence, it is steerable.

References

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