# Realization Theory for Discrete-Time Semi-Algebraic Hybrid Systems

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**Abstract.** We present realization theory for a class of autonomous discrete-time hybrid systems called *semi-algebraic hybrid systems*. These are systems in which the state and output equations associated with each discrete state are defined by polynomial equalities and inequalities. We first show that these systems generate the same output as semi-algebraic systems and implicit polynomial systems. We then derive necessary and almost sufficient conditions for existence of an implicit polynomial system realizing a given time-series data. We also provide a characterization of the dimension of a minimal realization as well as an algorithm for computing a realization from a given time-series data.

## 1 Introduction

Realization theory is one of the central topics of control and systems theory. Its goals are to study the conditions under which the observed behavior of a system can be represented by a state-space representation of a certain type and to develop algorithms for finding a (preferably minimal) state-space representation of the observed behavior. Realization theory forms the theoretical foundation of model reduction and systems identification. It also plays an important role in filtering and control design.

The goal of this paper is to develop realization theory and algorithms for the class of *autonomous discrete-time semi-algebraic hybrid systems*. Semi-algebraic hybrid systems (SAHSs) are characterized by the following two properties. First, the state and output trajectories are obtained by switching between various continuous subsystems. Second, the state-transition and output maps of each continuous subsystem are semi-algebraic functions, that is functions defined by polynomial equalities and inequalities. Particular examples of semi-algebraic functions are polynomial maps, piecewise-polynomial maps and piecewise-affine maps. The class of SAHSs includes important classes of discrete-time dynamical systems, such as *linear systems*, *polynomial systems*, and *piecewise-affine* hybrid systems. Furthermore, notice that semi-algebraic continuous state-transition maps can be used to encode discrete-state transition maps, semi-algebraic resets maps and guards. Hence, the class of SAHSs does implicitly allow for guards and resets. In this paper, we will deal only with *autonomous* SAHSs.

**Papers contributions.** We present a necessary condition for existence of an SAHS realization. The condition is formulated in terms of the finiteness of the (Krull) dimension of the algebra generated by the system outputs. We call this condition the *algebraic Hankel-rank* condition, as it is a natural generalization of the well-known Hankel-rank condition for linear systems. We show that the dimension of a minimal realization is bounded from below by the algebraic Hankel-rank. We also present an algorithm for computing an almost minimal SAHS realization from a given time-series data.

The results of the paper are based on the following behavioral relationships.

1. **Semi-algebraic hybrid systems $=$ semi-algebraic systems.** We will show that the output of an SAHS can be generated by a discrete-time system with semi-algebraic state-transition and output maps. The converse is trivially true.
2. **Semi-algebraic systems $\subseteq$ implicit polynomial systems.** We will show that the output of a dynamical system with semi-algebraic equations can be expressed as the output of a dynamical system defined by means of implicit polynomial equations.
3. **Implicit polynomial systems $\subseteq$ semi-algebraic hybrid systems**. We will show that the output of a dynamical system given by implicit polynomial equations can be generated by an SAHS. In fact, the switching signal of the hybrid system indicates which solution of the implicit polynomial equations should be chosen at each time.

By exploring the above relationships, we will be able to solve the realization problem for SAHSs by solving the realization problem for implicit polynomial systems. The solution of the latter problem is closely related to, and is inspired by, the work of Sontag [2] on discrete-time polynomial systems. The main difference with respect to [2] is that the algebras we work with are no longer integral domains.

The approach proposed in this paper bears a close resemblance to the algebraic-geometric approach to identification of switched autoregressive exogenous (SARX) systems of Vidal et al. [19–21]. In fact, the reduction of the realization problem for hybrid systems to finding implicit polynomial equations is analogous to the idea of the *hybrid decoupling polynomial* of [19–21]. The main differences lie in the classes of systems that are investigated and in the goals. The work of [19–21] investigates SARX systems and aims to obtain an SARX representation. Here we study systems which are autonomous, but otherwise more general than SARX systems, and aim to obtain a *more general* semi-algebraic hybrid system representation from the output data.

**Prior work.** The realization problem is well studied for deterministic and stochastic linear systems thanks to the works of Kalman and others (see e.g., [29, 30]). For bilinear and smooth/analytic nonlinear systems, the realization problem is also well understood thanks to the works of Sussmann, Jakubczyk, Sontag, Fliess, Isidori and others (see e.g., [1, 5–7, 2–4]). However, the algorithmic aspects of the theory are not fully developed for general nonlinear systems. There are important results on realization theory of polynomial and rational systems developed by Bartoszewicz, Sontag, Wang, etc., [8, 2, 9]. However, the study of minimality and realization algorithms is not well understood. The work of Grossmann and Larson [10] is one of the first attempts to tackle realization of hybrid systems. However, a formal realization theory is not presented. More recently, several papers have dealt with realization theory of switched linear/bilinear systems [11–13], linear/bilinear hybrid systems without guards and with partially observed discrete states [14, 13], nonlinear hybrid systems without guards [13, 15], piecewise-linear hybrid systems [16, 13], and stochastic jump-Markov linear systems [17, 18].

**Paper outline.** The paper is organized as follows. §2 presents the necessary algebraic preliminaries. §3 formulates the realization problem and states the main result of the paper formally. §4 contains the sketch of the proofs of the main results along with the realization algorithm. §5 presents the conclusions and directions for future work.

## 2 Algebraic Preliminaries

In this section we review some basic results from commutative algebra and semi-algebraic geometry. The reader is referred to [22–25] for more details. In particular, the reader is encouraged to consult [23, 22] for the definition and basic properties of Gröbner bases and Noether normalization. In what follows the term *algebra* denotes a commutative algebra over the field of real numbers $\mathbb{R}$, equipped with a unit element.

**Polynomials in finitely many commuting variables.** Let $A$ be an algebra. Recall from [22, 23] that $A[X_1, X_2, \dots, X_n]$ is the algebra of polynomials in the commuting variables $X_1, \dots, X_n$ over the algebra $A$. The elements of $A[X_1, X_2, \dots, X_n]$ are finite formal sums

$$P = \sum_{\alpha_1,\dots,\alpha_n \in I} a_{\alpha_1,\dots,\alpha_n} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n},$$

where $a_{\alpha_1,\dots,\alpha_n} \in A$ and $I$ is a finite set of natural numbers (possibly including zero). We will identify $X_i^0$ with the unit element $1$ of $A$ for all $i = 1, \dots, n$. If we want to emphasize the dependence of $P$ on the variables $X_1, X_2, \dots, X_n$, we will write $P(X_1, X_2, \dots, X_n)$ instead of $P$.

**Semi-algebraic sets and maps.** Recall from [24, 25] that a subset $S \subseteq \mathbb{R}^n$ is called *semi-algebraic* if it is of the form

$$S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \bigvee_{i=1}^{d} \bigwedge_{j=1}^{m_i} (P_{i,j}(x_1, \dots, x_n) \ \ \epsilon_{i,j} \ \ 0)\},$$

where for each $i = 1, \dots, d$ and $j = 1, \dots, m_i$ the symbol $\epsilon_{i,j}$ belongs to the set of symbols $\{<, >, \leq, \geq, =\}$ and $P_{i,j}$ is a polynomial in $\mathbb{R}[X_1, \dots, X_n]$. Here $\bigvee$ stands for the *logical or* operator and $\bigwedge$ stands for the *logical and* operator. Consider a subset $V$ of $\mathbb{R}^n$ and a map $f : V \to \mathbb{R}^m$. Recall from [24, 25] that the map $f$ is said to be a *semi-algebraic map*, if the graph of $f$ is a semi-algebraic set.

**Finitely generated algebra.** Let $A$ be an algebra and let $x_1, \dots, x_n \in A$. Denote by $\mathbb{R}[x_1, \dots, x_n]$ the smallest sub-algebra of $A$ which contains $x_1, \dots, x_n$. We will call $\mathbb{R}[x_1, \dots, x_n]$ the *algebra generated* by $x_1, \dots, x_n$. The algebra $A$ is called *finitely generated* if there exist finitely many elements $x_1, \dots, x_n$ of $A$ such that $A = \mathbb{R}[x_1, \dots, x_n]$.

**Krull-dimension of a finitely generated algebra.** Consider a finitely generated algebra $A = \mathbb{R}[x_1, \dots, x_n]$. Consider elements $z_1, \dots, z_d$ of $A$. We will say that $z_1, \dots, z_d$ are *algebraically independent*, if the only polynomial $Q \in \mathbb{R}[Z_1, \dots, Z_d]$ such that $Q(z_1, \dots, z_d) = 0$ is the zero polynomial. Here, $Q(z_1, \dots, z_n)$ is the element of $A$ obtained from $Q$ by substituting for each variable $Z_i$ the element $z_i$ and evaluating the resulting expression using the addition and multiplication operations in $A$. The Krull-dimension of $A$ is the *maximal number of algebraically independent elements of* $A$. We refer to the *Krull-dimension* of $A$ simply as the *dimension* of $A$ and denote it by $\dim A$.

**Algebra of time-series.** The algebra of time-series plays a crucial role in this paper. Consider the set $\mathbb{R}^\infty$ of all infinite sequences of real numbers. A typical element of $\mathbb{R}^\infty$ is of the form $(b(n))_{n \in \mathbb{N}}$, where $b(n) \in \mathbb{R}$ for all $n$. We will also refer to the elements of $\mathbb{R}^\infty$ as time-series, by interpreting a sequence as a sequence of measured system outputs. We define the addition and multiplication of time-series point-wise. That is, given two time-series $(a(n))_{n\in\mathbb{N}}$ and $(b(n))_{n\in\mathbb{N}}$, their sum is defined as the time-series $(a(n))_{n\in\mathbb{N}} + (b(n))_{n\in\mathbb{N}} = (a(n)+b(n))_{n\in\mathbb{N}}$, and their product is defined as the time-series $(a(n))_{n\in\mathbb{N}} \cdot (b(n))_{n\in\mathbb{N}} = (a(n)b(n))_{n\in\mathbb{N}}$. It is easy to see that, with the operations above, $\mathbb{R}^\infty$ forms an algebra. Its null element is the time-series in which every element is zero. Its identity element is the time-series where each element is 1. Moreover, each real number $x$ can be identified with the time-series where each element is equal to $x$.

## 3 Problem Formulation and Statement of the Main Results

The goals of this section are to define formally the notions of semi-algebraic systems (§3.1), semi-algebraic hybrid systems (§3.2) and implicit polynomial systems (§3.3), and to state the main results on realization theory and minimality for these classes of systems (§3.4). The proofs of these results together with a realization algorithm will be presented in the next section.

Before proceeding further, let us fix some notation and terminology. Throughout the paper we will look at discrete-time systems, i.e. our time axis will be the set of natural numbers including zero. We will denote the time axis by $\mathbb{N}$ and hence $0 \in \mathbb{N}$. Also, we will use $(\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}} \in \mathbb{R}^p$ to denote $\mathbb{R}^p$ valued time-series, i.e. $\widetilde{\mathbf{y}}(k) \in \mathbb{R}^p$, $k \in \mathbb{N}$. For each $i = 1, 2, \ldots, p$, we will denote by $\widetilde{\mathbf{y}}_i(k)$ the $i$th coordinate of the vector $\widetilde{\mathbf{y}}(k)$.

### 3.1 Semi-Algebraic Systems

A *semi-algebraic system* (SAS) is a discrete-time system of the form

$$\mathcal{S}_p : \begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k)), & \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y}(k) = h(\mathbf{x}(k)), \end{cases} \tag{1}$$

where for each $k \in \mathbb{N}$, the *state* $\mathbf{x}(k)$ at time $k$ belongs to $\mathbb{R}^n$ and the *output* $\mathbf{y}(k)$ at time $k$ belongs to $\mathbb{R}^p$. The *state-transition* map $f : \mathbb{R}^n \to \mathbb{R}^n$ and the *readout map* $h : \mathbb{R}^n \to \mathbb{R}^p$ are semi-algebraic maps. The state $\mathbf{x}_0$ is the *initial state* of the system. It is clear that the external behavior of (1) can be characterized by the time-series $(\mathbf{y}(k))_{k\in\mathbb{N}}$.

**Definition 1 (Realization by SASs)** *We will say that a system $\mathcal{S}_p$ of the form* (1) *is a realization of $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}} \in \mathbb{R}^p$ if for all time instants $k \in \mathbb{N}$, $\widetilde{\mathbf{y}}(k) = \mathbf{y}(k)$.*

We define the *dimension* of $\mathcal{S}_p$, denoted by $\dim \mathcal{S}_p$, as the number of state variables, i.e. $\dim \mathcal{S}_p = n$. Assume that $\mathcal{S}_p$ is a realization of a time-series $\mathcal{Y}$. We will say that $\mathcal{S}_p$ is a *minimal realization* of $\mathcal{Y}$ if $\mathcal{S}_p$ is a realization of $\mathcal{Y}$ that has the smallest possible dimension among all possible SASs that realize $\mathcal{Y}$.

### 3.2 Semi-Algebraic Hybrid Systems

A *semi-algebraic hybrid system* (SAHS) is a discrete-time hybrid (switched) system of the form

$$\mathcal{H}_p : \begin{cases} \mathbf{x}(k+1) = f_{q(k)}(\mathbf{x}(k)), & \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y}(k) = h_{q(k)}(\mathbf{x}(k)), \end{cases} \tag{2}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ denotes the *continuous state* at time $k \in \mathbb{N}$, $\mathbf{x}_0$ denotes the *initial state* of the system, $\mathbf{y}(k) \in \mathbb{R}^p$ denotes the *continuous output* at time $k \in \mathbb{N}$, and $q(k) \in Q$ denotes the *discrete mode* at time $k \in \mathbb{N}$. Here we assume that the set $Q$ is *finite*. The switching signal $(q(k))_{k\in\mathbb{N}}$ is assumed to be arbitrary. Also, for each discrete mode $q \in Q$, the maps $f_q : \mathbb{R}^n \to \mathbb{R}^n$ and $h_q : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be semi-algebraic, hence the name semi-algebraic hybrid systems. The definition of a realization for an SAHS is analogous to Definition 1.

**Definition 2 (Realization by SAHSs)** *An SAHS $\mathcal{H}_p$ of the form* (2) *is a realization of $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}} \in \mathbb{R}^p$ if for all $k \in \mathbb{N}$, $\widetilde{\mathbf{y}}(k) = \mathbf{y}(k)$.*

We will call the number continuous state variables $n$ the *dimension* of $\mathcal{H}_p$, and we will denote it by $\dim \mathcal{H}_p$, i.e. $\dim \mathcal{H}_p = n$. We will call an SAHS $\mathcal{H}_p$ a *minimal* realization of $\mathcal{Y}$ if $\mathcal{H}_p$ is a realization of $\mathcal{Y}$ with the smallest dimension among all possible SAHS realizations of $\mathcal{Y}$. One may wonder whether this definition of minimality is justified, as it does not take into the account the number of discrete modes. We think this is an interesting direction to explore. However, we are not aware of any work in this direction.

### 3.3 Implicit Polynomial Systems

An *implicit polynomial system* (IPS) is a discrete-time dynamical system of the form

$$\mathcal{P}_p : \begin{cases} Q_i(\mathbf{x}_i(k+1), \mathbf{x}_1(k), \ldots, \mathbf{x}_n(k)) = 0 \text{ for all } i = 1, \ldots, n \\ P_j(\mathbf{y}_j(k), \mathbf{x}_1(k), \ldots, \mathbf{x}_n(k)) = 0 \text{ for all } j = 1, \ldots, p. \end{cases} \tag{3}$$

In the above equation, $\mathbf{x}(k) = (\mathbf{x}_1(k), \ldots, \mathbf{x}_n(k))^\top \in \mathbb{R}^n$ is the *continuous state* at time $k \in \mathbb{N}$, $\mathbf{y}(k) = (\mathbf{y}_1(k), \mathbf{y}_2(k), \ldots, \mathbf{y}_p(k))^\top \in \mathbb{R}^p$ is the *continuous output* at time $k \in \mathbb{N}$, $\mathbf{x}(0) = \mathbf{x}_0$ is the *initial state* of the system, and for each $i = 1, \ldots, n$ and $j = 1, \ldots, p$, $Q_i(Z_0, Z_1, \ldots, Z_n)$ and $P_j(Z_0, Z_1, \ldots, Z_n)$ are polynomials in the variables $Z_0, \ldots, Z_n$ with real coefficients. In addition, we will assume the following.

**Assumption 1** *For all $k \in \mathbb{N}$, $i = 1, \ldots, n$, and $j = 1, \ldots, p$, $P_j(Z_0, \mathbf{x}_1(k), \ldots, \mathbf{x}_n(k))$ and $Q_i(Z_0, \mathbf{x}_1(k), \ldots, \mathbf{x}_n(k))$ are non-zero polynomials in $Z_0$.*

If the assumption above fails for some $k$, then one of the components of $\mathbf{y}(k)$ or $\mathbf{x}(k+1)$ can be chosen indepently of the state $\mathbf{x}(k)$.

Notice that the state and output of (3) at time $k$ are not determined solely by the initial state $\mathbf{x}(0) = \mathbf{x}_0$. The reason for this is that the current state determines the next state and the current output *implicitly*, and hence several valid choices for the output and next state may exist. In the sequel, whenever we speak of an IPS of the form (3), we will always assume that a specific state trajectory $(\mathbf{x}(k))_{k\in\mathbb{N}}$ and output trajectory $(\mathbf{y}(k))_{k\in\mathbb{N}}$ is fixed, such that $(\mathbf{x}(k))_{k\in\mathbb{N}}$ and $(\mathbf{y}(k))_{k\in\mathbb{N}}$ satisfy (3).

**Definition 3 (Realization by IPSs)** *An IPS $\mathcal{P}_p$ of the form (3) with state trajectory $(\mathbf{x}(k))_{k\in\mathbb{N}} \in \mathbb{R}^n$ and output trajectory $(\mathbf{y}(k))_{k\in\mathbb{N}} \in \mathbb{R}^p$ is said to be a realization of the time-series $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}} \in \mathbb{R}^p$ if for all $k \in \mathbb{N}$, $\widetilde{\mathbf{y}}(k) = \mathbf{y}(k)$.*

As before, we define the *dimension* of an IPS $\mathcal{P}_p$ of the form (3), denoted by $\dim \mathcal{P}_p$, to be the number of state variables, i.e. $\dim \mathcal{P}_p = n$. An IPS $\mathcal{P}_p$ is said to be a *minimal realization* of $\mathcal{Y}$ if $\mathcal{P}_p$ is a realization of $\mathcal{Y}$ that has the smallest dimension among all possible IPSs that realize $\mathcal{Y}$.

### 3.4 Main Results

In what follows, we state the main results of the paper on realization of SASs, SAHSs, and IPSs. We begin with Theorem 1, which states the main result on output equivalence of these systems. Then in Theorems 2–4 we state the main results on existence and minimality of realizations. The proof of Theorem 1 (see §4.1) yields a number of procedures for converting systems from one of these classes to the others. Before stating the theorem formally, we need to introduce some notation for each one of these transformations.

**Notation 1** *The proof of Theorem 1 yields the following transformations.*

***Procedure for transforming SAHSs to SASs.*** *Given an SAHS $\mathcal{H}_p$, we will denote by $SA(\mathcal{H}_p)$ the SAS which is the outcome of this procedure if applied to $\mathcal{H}_p$.*
***Procedure for transforming SASs to IPSs.*** *Given an SAS $\mathcal{S}_p$, we will denote by $IP(\mathcal{S}_p)$ the IPS which is the outcome of this procedure if applied to $\mathcal{S}_p$.*
***Procedure for transforming IPSs to SAHSs.*** *Given an IPS $\mathcal{P}_p$, we will denote by $SAH(\mathcal{P}_p)$ the SAH which is the outcome of this procedure if applied to $\mathcal{P}_p$.*

With this notation, we are ready to state the main result on equivalence of the output behaviors generated by these systems.

**Theorem 1 (Equivalence of SASs, SAHSs and IPSs)** *Let $\mathcal{S}_p$ be an SAS, $\mathcal{H}_p$ be an SAHS, and $\mathcal{P}_p$ be an IPS satisfying Assumption 1. Let $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}} \in \mathbb{R}^p$ be a time-series. Then the following holds.*

- *$\mathcal{H}_p$ is a realization of $\mathcal{Y}$ if and only if $SA(\mathcal{H}_p)$ is a realization of $\mathcal{Y}$. In addition, $\dim SA(\mathcal{H}_p) = \dim \mathcal{H}_p + 1$.*
- *$\mathcal{S}_p$ is a realization of $\mathcal{Y}$ if and only if $IP(\mathcal{S}_p)$ is a realization of $\mathcal{Y}$. In addition, $\dim IP(\mathcal{S}_p) = \dim \mathcal{S}_p + 1$ and $IP(\mathcal{S}_p)$ satisfies Assumption 1.*
- *If $\mathcal{P}_p$ is a realization of $\mathcal{Y}$, then $SAH(\mathcal{P}_p)$ is a realization of $\mathcal{Y}$. In addition, $\dim SAH(\mathcal{P}_p) = \dim \mathcal{P}_p$.*

We now state the main result on existence of a realization by an IPS, and hence by an SAHS or SAS. To that end, recall from linear systems theory the definition of the Hankel-matrix $H_\mathcal{Y}$ associated with the time-series $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}} \in \mathbb{R}^p$. The matrix $H_\mathcal{Y} \in \mathbb{R}^{\infty\times\infty}$ has an infinite number of rows and columns indexed by natural numbers, and the entry of $H_\mathcal{Y}$ indexed by $((l-1)p+i, j)$ with $j, l = 1, 2, 3, \ldots$, and $i = 1, \ldots, p$ equals $\widetilde{\mathbf{y}}_i(l+j-2)$. Let $H_{\mathcal{Y},N} \in \mathbb{R}^{pN\times\infty}$ be the matrix formed by all rows of $H_\mathcal{Y}$ indexed by indices of the form $k = lp+i$ with $l = 0, \ldots, N$ and $i = 1, \ldots, p$. That is, $H_{\mathcal{Y},N}$ is of the form

$$H_{\mathcal{Y},N} = \begin{bmatrix} \widetilde{\mathbf{y}}_1(0) & \widetilde{\mathbf{y}}_1(1) & \cdots & \widetilde{\mathbf{y}}_1(j) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\mathbf{y}}_i(l) & \widetilde{\mathbf{y}}_i(l+1) & \cdots & \widetilde{\mathbf{y}}_i(l+j) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\mathbf{y}}_p(N) & \widetilde{\mathbf{y}}_p(N+1) & \cdots & \widetilde{\mathbf{y}}_p(N+j) & \cdots \end{bmatrix}. \tag{4}$$

A classical result from linear systems theory is that the time-series $\mathcal{Y}$ admits an autonomous linear system realization if and only if the rank of the Hankel-matrix $H_{\mathcal{Y}}$ is finite, or equivalently, there is an upper bound on the ranks of the set of matrices $\{H_{\mathcal{Y},N}, N \in \mathbb{N}\}$. Below we will extend this well-known finite Hankel-rank condition to IPSs, by introducing the notion of *algebraic rank* of $H_{\mathcal{Y}}$.

**Definition 4 (Hankel-algebra)** *Define the sub-algebra $\mathcal{A}_{\mathcal{Y},N}$ of $\mathbb{R}^{\infty}$ as the sub-algebra generated by the rows of the matrix $H_{\mathcal{Y},N}$ viewed as scalar time-series. We will call the sub-algebra $\mathcal{A}_{\mathcal{Y},N}$ the $N$*-Hankel-algebra of $\widetilde{\mathbf{y}}$.

**Definition 5 (Algebraic rank of the Hankel-matrix)** *Define the algebraic rank of the Hankel-matrix $H_{\mathcal{Y}}$, denoted by alg-rank $H_{\mathcal{Y}}$, as the supremum of the Krull-dimensions of the $N$-Hankel-algebras. That is,*

$$\textit{alg-rank } H_{\mathcal{Y}} = \sup_{N \in \mathbb{N}} \dim \mathcal{A}_{\mathcal{Y},N}. \quad (5)$$

**Remark 1 (Finite rank of the Hankel-matrix implies finite algebraic rank)** *Notice that if the rank of the Hankel-matrix is finite, then its algebraic rank is also finite.*

**Theorem 2 (Existence and minimality of an IPS realization)** *A time-series $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k \in \mathbb{N}} \in \mathbb{R}^p$ has a realization by an IPS satisfying Assumption 1 only if the algebraic rank of the Hankel-matrix $H_{\mathcal{Y}}$ is finite. In addition, the dimension of any IPS realization $\mathcal{P}_p$ of $\mathcal{Y}$ satisfying Assumption 1 is at least alg-rank $H_{\mathcal{Y}}$. Moreover, if alg-rank $H_{\mathcal{Y}} = n < +\infty$, then we can construct an IPS realization of $\mathcal{Y}$ whose dimension is $n$, but which does not necessarily satisfy Assumption 1.*

We will say that an SAHS $\mathcal{H}_p$ is an *almost minimal* realization of $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k \in \mathbb{N}}$ if $\dim \mathcal{H}_p =$ alg-rank $H_{\mathcal{Y}}$ and $\mathcal{H}_p$ is a realization of $\mathcal{Y}$. We will say that an SAS $\mathcal{S}_p$ is an *almost minimal* realization of $\mathcal{Y}$ if $\mathcal{S}_p$ is a realization of $\mathcal{Y}$ and $\dim \mathcal{S}_p =$ alg-rank $H_{\mathcal{Y}} + 1$. Combining Theorem 2 with Theorem 1 we get the following realization theorems.

**Theorem 3 (Existence and minimality of an SAHS realization)** *A time-series $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k \in \mathbb{N}} \in \mathbb{R}^p$ has a realization by an SAHS only if alg-rank $H_{\mathcal{Y}} < +\infty$. In addition, the dimension of a minimal SAHS realization of $\mathcal{Y}$ is at least alg-rank $H_{\mathcal{Y}} - 2$. Moreover, if $\mathcal{Y}$ admits an IPS realization $\mathcal{P}_p$ such that $\dim \mathcal{P}_p =$ alg-rank $H_{\mathcal{Y}}$ and $\mathcal{P}_p$ satisfies Assumption 1, then $SAH(\mathcal{P}_p)$ is an almost minimal SAHS realization of $\mathcal{Y}$.*

**Theorem 4 (Existence and minimality of an SAS realization)** *A time-series $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k \in \mathbb{N}} \in \mathbb{R}^p$ has a realization by an SAS only if alg-rank $H_{\mathcal{Y}} < +\infty$. In addition, the dimension of a minimal SAS realization of $\mathcal{Y}$ is at least alg-rank $H_{\mathcal{Y}} - 1$. Moreover, if $\mathcal{Y}$ has an IPS realization $\mathcal{P}_p$ such that $\dim \mathcal{P}_p =$ alg-rank $H_{\mathcal{Y}}$ and $\mathcal{P}_p$ satisfies Assumption 1, then $SA(SAH(\mathcal{P}_p))$ is an almost minimal SAS realization of $\mathcal{Y}$.*

Theorems 2-4 establish conditions for existence and minimality of IPS, SAHS, and SAS realizations of $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k \in \mathbb{N}} \in \mathbb{R}^p$. In §4.3, we will show that under suitable assumptions one can actually construct a minimal IPS realization $\mathcal{P}_p$ from the rows of $H_{\mathcal{Y}}$. Furthermore, we will show that one can use $\mathcal{P}_p$ to construct an almost minimal SAHS realization of $\mathcal{Y}$ and an almost minimal SAS realization of $\mathcal{Y}$. Before delving into the details of these constructions, together with the corresponding realization algorithms, we shall provide in §4.1-§4.2 the proofs for Theorems 1-4.

## 4 Realization Construction

In this section, we sketch the constructions that lie at the heart of the proofs of Theorems 1–4. In §4.1 we present the proof of Theorem 1. In §4.2 we present the proof of Theorems 2–4. Finally, in §4.3 we discuss the algorithmic aspects of realization theory.

### 4.1 Proof of Theorem 1

The proof of Theorem 1 will be divided into the following three parts.

**Definition of $SA(\mathcal{H}_p)$ and its properties.** Consider an SAHS $\mathcal{H}_p$ of the form (2) and let $w = (q(k))_{k\in\mathbb{N}} \in Q$ be its switching signal. Since the set of discrete modes $Q$ is finite, we can assume without loss of generality that $Q$ is of the form $Q = \{1, 2, \dots, d\}$. As shown in [16, 27], this allows one to encode the switching signal $w$ as a real number in the interval $[0, 1]$ by using the following procedure. Define the *encoding* $\psi(w)$ of $w$ as $\psi(w) = \sum_{k=0}^{\infty} \frac{(q(k)-1)}{(2d)^{k+1}}$. It is easy to see that this series is absolutely convergent and that $0 \le \psi(w) < 1$. Recall also from [16, 27] that there exist piecewise-affine operations $H : [0, 1] \to \mathbb{R}$ and $M : [0, 1] \to [0, 1]$ such that $H(\psi(w)) = q(0)$ and $M(\psi(w)) = \psi((q(k+1))_{k\in\mathbb{N}})$. That is, $H(\psi(w))$ returns the first element of the sequence $w$, and $M(\psi(w))$ returns the encoding of the *shift* of $w$. For each $z \in [0, 1]$, these operations can be written explicitly as:

$$
\begin{aligned}
H(z) &= \begin{cases} i+1 & \text{if } i \le 2dz < i+1 \text{ for some } i = 0,\dots,d-1 \\ d & \text{otherwise} \end{cases} \\
M(z) &= \begin{cases} 2dz - i & \text{if } i \le 2dz < i+1 \text{ for some } i = 0,\dots,d-1 \\ z & \text{otherwise} \end{cases} .
\end{aligned} \tag{6}
$$

Furthermore, it is easy to see that $H$ and $M$ can be extended to piecewise-affine maps defined on the whole $\mathbb{R}$. We can then obtain $SA(\mathcal{H}_p)$ from $\mathcal{H}_p$ by adding a new state variable $\mathbf{z}(k)$ that equals the encoding $\psi((q(l+k))_{l\in\mathbb{N}})$ of the future switching sequence. That is

$$
SA(\mathcal{H}_p) : \begin{cases} \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{z}(k+1) \end{bmatrix} = \begin{bmatrix} \widetilde{f}(\mathbf{x}(k), \mathbf{z}(k)) \\ M(\mathbf{z}(k)) \end{bmatrix} \\ \mathbf{y}(k) = \widetilde{h}(\mathbf{x}(k), \mathbf{z}(k)) \end{cases} \tag{7}
$$

where $\mathbf{x}(0) = \mathbf{x}_0$ coincides with the initial state of $\mathcal{H}_p$ and $\mathbf{z}(0) = \mathbf{z}_0 = \psi(w)$, and the maps $\widetilde{f}$ and $\widetilde{h}$ are defined as

$$
\begin{aligned}
\widetilde{f}(x, z) &= \begin{cases} f_q(x) & \text{if } H(z) = q \text{ for some } q \in Q \\ f_d(x) & \text{otherwise} \end{cases} \\
\widetilde{h}(x, z) &= \begin{cases} h_q(x) & \text{if } H(z) = q \text{ for some } q \in Q \\ h_d(x) & \text{otherwise} \end{cases}
\end{aligned} \tag{8}
$$

It is easy to see that $\widetilde{f}$ and $\widetilde{h}$ are semi-algebraic maps. It is also easy to see that $\mathbf{y}(k)$ and $\mathbf{x}(k)$ in (7) are the same as $\mathbf{y}(k)$ and $\mathbf{x}(k)$ in (2) for all time instances $k \in \mathbb{N}$. Hence, the system in (7) is a well-defined SAS. Furthermore, it is a realization of $(\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}}$ if and only if $\mathcal{H}_p$ is a realization of $(\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}}$, and $\dim SA(\mathcal{H}_p) = \dim \mathcal{H}_p + 1$.

**Definition of $IP(\mathcal{S}_p)$ and its properties.** Consider an SAS $\mathcal{S}_p$ of the form (1) with state transition map $f : \mathbb{R}^n \to \mathbb{R}^n$ and readout map $h : \mathbb{R}^n \to \mathbb{R}^p$. For all $i = 1, \ldots, n$ and $j = 1, \ldots, p$, denote by $f_i : \mathbb{R}^n \ni x \mapsto f_i(x) \in \mathbb{R}$ and $h_j : \mathbb{R}^n \ni x \mapsto h_j(x) \in \mathbb{R}$ the semi-algebraic maps obtained from the $i$th and $j$th coordinates of $f$ and $h$, respectively. It follows from the proof of Proposition 8.13.7 in [24] that there exist polynomials in $\mathbb{R}[Z_0, \ldots, Z_{n+1}]$, $\{Q_i(Z_0, \ldots, Z_n, Z_{n+1})\}_{i=1}^n$ and $\{P_j(Z_0, \ldots, Z_n, Z_{n+1})\}_{j=1}^p$, such that the following holds: There exists a *finite* subset of $\mathbb{R}$, $\mathcal{D} = \{d_1, \ldots, d_M\} \subseteq \mathbb{R}$, such that for all $x_1, \ldots, x_n \in \mathbb{R}$ there exists $\gamma = \gamma(x_1, \ldots, x_n) \in \mathcal{D}$ such that $P_j(Z_0, x_1, \ldots, x_n, \gamma)$ and $Q_i(Z_0, x_1, \ldots, x_n, \gamma)$ are nonzero polynomials in $Z_0$, and

$$Q_i(f_i(x), x_1, \ldots, x_n, \gamma) = 0 \text{ and } P_j(h_j(x), x_1, \ldots, x_n, \gamma) = 0$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, p$. We can then define $IP(\mathcal{S}_p)$ as

$$\begin{aligned} Q_i(\mathbf{x}_i(k+1), \mathbf{x}_1(k), \ldots, \mathbf{x}_{n+1}(k)) = 0 \text{ for all } i = 1, \ldots, n+1 \\ P_j(\mathbf{y}_j(k), \mathbf{x}_1(k), \ldots, \mathbf{x}_{n+1}(k)) = 0 \text{ for all } j = 1, \ldots, p \end{aligned} \tag{9}$$

where the polynomials $Q_i, P_j$ for $i = 1, \ldots, n$, $j = 1, \ldots, p$ are as defined above and $Q_{n+1}(Z_0, \ldots, Z_{n+1}) = \Pi_{l=1}^M (Z_0 - d_l)$. The first $n$ state components $\mathbf{x}_1(k), \ldots, \mathbf{x}_n(k)$ of $IP(\mathcal{S}_p)$ coincide with those of $\mathcal{S}_p$. The $n+1$st state is defined as $\mathbf{x}_{n+1}(k) = \gamma(\mathbf{x}_1(k), \ldots, \mathbf{x}_n(k)) \in \mathcal{D}$. The output trajectory of $IP(\mathcal{S}_p)$ is the same as that of $\mathcal{S}_p$. It follows that $IP(\mathcal{S}_p)$ is a well defined IPS satisfying Assumption 1. Moreover, $\mathcal{S}_p$ is a realization of $\mathcal{Y}$ if and only if $IP(\mathcal{S}_p)$ is a realization of $\mathcal{Y}$, and $\dim IP(\mathcal{S}_p) = \dim \mathcal{S}_p + 1$.

**Definition of $SAH(\mathcal{P}_p)$ and its properties.** Let $\mathcal{P}_p$ be an IPS of the form (3) satisfying Assumption 1. Recall that by specifying $\mathcal{P}_p$ we fix a state-trajectory $(\mathbf{x}(k))_{k \in \mathbb{N}}$ and an output-trajectory $(\mathbf{y}(k))_{k \in \mathbb{N}}$ satisfying the equations in (3). Let $d_i$ and $r_j$ be, respectively, the degrees of the polynomials $Q_i(Z_0, Z_1, \ldots, Z_n)$ and $P_j(Z_0, Z_1, \ldots, Z_n)$ with respect to $Z_0$, for $i = 1, \ldots, n$, $j = 1, \ldots, p$. It follows from Proposition A.5 in [24] that there are semi-algebraic functions from $\mathbb{R}^n$ to $\mathbb{R}$, $\psi_{j,1}, \ldots, \psi_{j,r_j}$ and $\chi_{i,1}, \ldots, \chi_{i,d_i}$, $i = 1, \ldots, n$, $j = 1, \ldots, p$, such that for all $x_1, \ldots, x_n \in \mathbb{R}$, $Q_i(Z_0, x_1, \ldots, x_n)$ and $P_j(Z_0, x_1, \ldots, x_n)$ are non-zero polynomials over $Z_0$; if $Q_i(z, x_1, \ldots, x_n) = 0$, then $z = \chi_{i,l}(x_1, \ldots, x_n)$ for a unique $l = 1, \ldots, d_i$, and if $P_j(z, x_1, \ldots, x_n) = 0$, then $z = \psi_{j,k}(x_1, \ldots, x_n)$ for a unique $k = 1, \ldots, r_j$. We can then define $SAH(\mathcal{P}_p)$ as in (2), with the system parameters defined as follows. Let the set of discrete modes of $SAH(\mathcal{P}_p)$ be the set $Q$ of all $n+p$ tuples $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_n)$, where $\alpha_j = 1, \ldots, r_j$, and $\beta_i = 1, \ldots, d_i$, for all $j = 1, \ldots, p, i = 1, \ldots, n$. For each discrete mode $q \in Q$ of the form $q = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_n)$ define

$$\begin{aligned} f_q(x_1, \ldots, x_n) &= \left[\chi_{1,\beta_1}(x_1, \ldots, x_n) \; \chi_{2,\beta_2}(x_1, \ldots, x_n) \cdots \; \chi_{n,\beta_n}(x_1, \ldots, x_n)\right]^\top \\ h_q(x_1, \ldots, x_n) &= \left[\psi_{1,\alpha_1}(x_1, \ldots, x_n) \; \psi_{2,\alpha_2}(x_1, \ldots, x_n) \cdots \; \psi_{p,\alpha_p}(x_1, \ldots, x_n)\right]^\top \end{aligned} \tag{10}$$

It is easy to see that $f_q$ and $h_q$ are semi-algebraic functions for all discrete modes $q \in Q$. It is left to define the initial state and the switching signal of $SAH(\mathcal{P}_p)$. Recall that $(\mathbf{x}(k))_{k \in \mathbb{N}}$ and $(\mathbf{y}(k))_{k \in \mathbb{N}}$ are, respectively, the state and output trajectory of $\mathcal{P}_p$. It follows from the discussion above and Assumption 1 that for each time instant $k \in \mathbb{N}$ there exist indices $\beta_i(k) \in \{1, \ldots, d_i\}$, $i = 1, \ldots, n$ and $\alpha_j(k) \in \{1, \ldots, r_j\}$,

$j = 1, \ldots, p$, such that $\mathbf{x}_i(k+1)$ equals $\chi_{i,\beta_i(k)}(\mathbf{x}_1(k), \ldots, \mathbf{x}_n(k))$ and $\mathbf{y}_j(k)$ equals $\psi_{j,\alpha_j(k)}(\mathbf{x}_1(k), \ldots, \mathbf{x}_n(k))$. Choose the switching signal $w = (q(k))_{k \in \mathbb{N}}$ as $q(k) = (\alpha_1(k), \ldots, \alpha_p(k), \beta_1(k), \ldots, \beta_n(k)) \in Q$ and the initial state as $\mathbf{x}(0) = \mathbf{x}_0$. We get that $(\mathbf{x}(k))_{k \in \mathbb{N}}$ and $(\mathbf{y}(k))_{k \in \mathbb{N}}$ are the state and output trajectories of $SAH(\mathcal{P}_p)$. In particular, this implies that $SAH(\mathcal{P}_p)$ is a realization of $(\tilde{\mathbf{y}}(k))_{k \in \mathbb{N}}$. It is easy to see from the construction of $SAH(\mathcal{P}_p)$ that $\dim SAH(\mathcal{P}_p) = \dim \mathcal{P}_p$.

### 4.2 Proof of Theorems 2-4

Theorems 3 and 4 follow easily from Theorems 1 and 2. Therefore, it is enough to prove Theorem 2. We divide the proof of Theorem 2 into the following three parts.

**Necessity.** Assume that $\mathcal{Y} = (\tilde{\mathbf{y}}(k))_{k \in \mathbb{N}}$ has an IPS realization $\mathcal{P}_p$ of the form (3) satisfying Assumption 1. Notice that the time-series $(\mathbf{x}_i(k))_{k \in \mathbb{N}} \in \mathbb{R}$, $i = 1, \ldots, n$, formed by the components of the state trajectory belong to $\mathbb{R}^\infty$. In addition, for each $j = 1, \ldots, p$ the time-series $(\tilde{\mathbf{y}}_j(k))_{k \in \mathbb{N}} \in \mathbb{R}$, coincides with the time-series $(\mathbf{y}_j(k))_{k \in \mathbb{N}}$ formed by the $j$th coordinates of the output trajectory of $\mathcal{P}_p$. For each $N$ denote by $\mathcal{B}_{\mathcal{P}_p,N}$ the sub-algebra of $\mathbb{R}^\infty$ generated by the rows of $H_{\mathcal{Y},N}$ and by the time-series $(\mathbf{x}_i(k+l))_{k \in \mathbb{N}}$, $i = 1, \ldots, n$ and $l = 0, \ldots, N$. It is easy to see that the $N$-Hankel-algebra $\mathcal{A}_{\mathcal{Y},N}$ is a sub-algebra of $\mathcal{B}_{\mathcal{P}_p,N}$. Moreover, using Corollary 3.7 of [22] we see that for each $N$, $\dim \mathcal{A}_{\mathcal{Y},N} \leq \dim \mathcal{B}_{\mathcal{P}_p,N}$. If we can show that $\dim \mathcal{B}_{\mathcal{P}_p,N} \leq n$, then it follows that alg-rank $H_\mathcal{Y} \leq n < +\infty$. To that end, consider any minimal prime ideal $P$ of $\mathcal{B}_{\mathcal{P}_p,N}$ (see [22] for the definition of a minimal prime ideal of an algebra) and the sub-algebra $\mathcal{A}_x = \mathbb{R}[(\mathbf{x}_1(k))_{k \in \mathbb{N}}, \ldots, (\mathbf{x}_n(k))_{k \in \mathbb{N}}]$ of $\mathcal{B}_{\mathcal{P}_p,N}$. Using Assumption 1 it can be shown that $\mathcal{B}_{\mathcal{P}_p,N}/P$ is algebraic over $\mathcal{A}_x/(\mathcal{A}_x \cap P)$, and hence $\dim \mathcal{B}_{\mathcal{P}_p,N}/P \leq n$ for any minimal prime $P$. Since $\dim \mathcal{B}_{\mathcal{P}_p,N} = \max\{\dim \mathcal{B}_{\mathcal{P}_p,N}/P \mid P \text{ is a minimal prime}\}$ we get that $\dim \mathcal{B}_{\mathcal{P}_p,N} \leq n$.

**Sufficiency.** Assume that alg-rank $H_\mathcal{Y} = n < +\infty$. It follows that there exists $N^*$ such that for all $k > 0$, $n = \dim \mathcal{A}_{\mathcal{Y},N^*} = \dim \mathcal{A}_{\mathcal{Y},N^*+k}$. Choose a Noether Normalization (see [22]) $(\mathbf{z}_i(k))_{k \in \mathbb{N}} \in \mathbb{R}$, $i = 1, \ldots, n$, of $\mathcal{A}_{\mathcal{Y},N^*}$. Then the time-series $(\mathbf{z}_1(k))_{k \in \mathbb{N}}, \ldots, (\mathbf{z}_n(k))_{k \in \mathbb{N}}$ are algebraically independent and $\mathcal{A}_{\mathcal{Y},N^*+1}$ is algebraic over the algebra $\mathbb{R}[(\mathbf{z}_1(k))_{k \in \mathbb{N}}, \ldots, (\mathbf{z}_n(k))_{k \in \mathbb{N}}]$. Therefore, there exist polynomials $Q_i(T_0, Z_1, \ldots, Z_n)$ and $P_j(T_0, Z_1, \ldots, Z_n)$, $i = 1, \ldots, n$, $j = 1, \ldots, p$ such that

$$
\begin{aligned}
Q_i(\mathbf{z}_i(k+1), \mathbf{z}_1(k), \ldots, \mathbf{z}_n(k)) = 0 \text{ for all } i = 1, \ldots, n, k \in \mathbb{N} \\
P_j(\tilde{\mathbf{y}}_j(k), \mathbf{z}_1(k), \ldots, \mathbf{z}_n(k)) = 0 \text{ for all } j = 1, \ldots, p, k \in \mathbb{N}.
\end{aligned}
\tag{11}
$$

It is then easy to see that (11) defines an IPS realization of $\mathcal{Y}$ with the state trajectory $(\mathbf{z}(k))_{k \in \mathbb{N}}$, $\mathbf{z}(k) = (\mathbf{z}_1(k), \ldots, \mathbf{z}_n(k)) \in \mathbb{R}^n$, $k \in \mathbb{N}$, and output trajectory $(\tilde{\mathbf{y}}(k))_{k \in \mathbb{N}} \in \mathbb{R}^p$. We will call this IPS the *free realization* of $\mathcal{Y}$ and we will denote it by $\mathcal{P}_{\tilde{\mathbf{y}}}$. Notice that $\mathcal{P}_{\tilde{\mathbf{y}}}$ need not satisfy Assumption 1.

**Minimality.** The proof of the statement of Theorem 2 is now rather simple. First, from the proof of necessity of the finite algebraic rank of the Hankel-matrix, it follows that if $\mathcal{P}_p$ is an IPS realization of $(\tilde{\mathbf{y}}(k))_{k \in \mathbb{N}}$ and $\mathcal{P}_p$ satisfies Assumption 1, then alg-rank $H_\mathcal{Y} \leq \dim \mathcal{P}_p$. From the proof of sufficiency it follows that the free realization $\mathcal{P}_{\tilde{\mathbf{y}}}$ is an IPS realization of $(\tilde{\mathbf{y}}(k))_{k \in \mathbb{N}}$ and $\dim \mathcal{P}_{\tilde{\mathbf{y}}} =$ alg-rank $H_\mathcal{Y}$.

### 4.3 Realization Algorithms

In this section, we present realization algorithms for constructing an almost minimal IPS, SAS and SAHS realization of a time series. We first present a realization algorithm that returns the polynomials of an IPS realization $\mathcal{P}_p$ of the measured data along with a finite portion of the state trajectory. We then discuss how to use this algorithm for computing a minimal SAHS and SAS realization of the same series.

Throughout the section we will assume that the first $2M$ elements of the time series $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}}$ are measured for some $M \in \mathbb{N}$.

**Realization algorithm for IPSs.** The main idea behind the realization algorithm we are about to present is that each Hankel-algebra $\mathcal{A}_{\mathcal{Y},N}$, $N \in \mathbb{N}$, can be represented as a quotient of a polynomial ring with a suitable ideal $I_N$. Then, given a Gröbner basis for $I_N$, the computation of the polynomials defining $\mathcal{P}_p$ can be done using Gröbner-basis techniques. The following paragraphs describe the algorithm in more detail.

For each $N$, let $\mathbb{R}[\mathcal{T}_N]$ be the ring of polynomials $\mathbb{R}[T_1, \ldots, T_{(N+1)p}]$ in the variables $T_1, \ldots, T_{(N+1)p}$. Also let $I_N$ be the ideal of $\mathbb{R}[\mathcal{T}_N]$ generated by all the polynomials that vanish on the set

$$V_N = \{(\widetilde{\mathbf{y}}(k)^\top, \ldots, \widetilde{\mathbf{y}}(k+N)^\top)^\top \in \mathbb{R}^{p(N+1)} \mid k \in \mathbb{N}\}. \quad (12)$$

Then, it is easy to see that $\mathcal{A}_{\mathcal{Y},N}$ is isomorphic to the quotient $\mathcal{A}_{\mathcal{Y},N} \cong \mathbb{R}[\mathcal{T}_N]/I_N$. Denote by $G_N$ the Gröbner-basis of $I_N$. Choose a number $D > 0$ representing our guess on the maximal degree of polynomials generating the ideals $I_N$. We are now ready to formulate the partial realization algorithm $\mathrm{IPPartReal}(M, D)$ for IPSs.

---

$\mathrm{IPPartReal}(M, D)$

---

1: Set $N := 0$.
2: Compute the Gröbner basis of $I_N$ and $I_{N+1}$
$G_N := \mathrm{ApproxIdeal}(M, D, N)$, $G_{N+1} := \mathrm{ApproxIdeal}(M, D, N+1)$.
3: Compute the Noether Normalization of $G_N$ and $G_{N+1}$
$(\{Y_1^l, \ldots, Y_{d_l}^l\}, d_l) = \mathrm{NoetherNorm}(l, G_l)$ for $l = N, N+1$.
4: If $(d_{N+1} > d_N)$ and $(N+2 \leq M)$, then go back to Step 2 with $N := N+1$.
5: Compute the polynomials of the free IPS realization $\mathcal{P}_{\widetilde{\mathbf{y}}}$ as follows.
Let $d := d_N = d_{N+1}$.
For each $i = 1, \ldots, d$, let $Z_i(T_1, \ldots, T_{(N+2)p}) := Y_i^N(T_{p+1}, T_{p+2}, \ldots, T_{(N+2)p})$.
For each $i = 1, \ldots, d$, let $Q_i := \mathrm{DepPoly}(N+1, Y_1^N, \ldots, Y_d^N, Z_i, G_{N+1})$.
For each $j = 1, \ldots, p$, let $P_j := \mathrm{DepPoly}(N+1, Y_1^N, \ldots, Y_d^N, T_j, G_{N+1})$.
For each $i = 1, \ldots, d$, define $\mathbf{z}_i(k) := Y_i^N((\widetilde{\mathbf{y}}(k)^\top, \ldots, \widetilde{\mathbf{y}}(k+N)^\top)^\top)$.
6: Return the IPS $\mathcal{P}_{\widetilde{\mathbf{y}}}$ defined as

$$\mathcal{P}_{\widetilde{\mathbf{y}}} \begin{cases} Q_i(\mathbf{z}_i(k+1), \mathbf{z}_1(k), \ldots, \mathbf{z}_d(k)) = 0 \text{ for all } i = 1, \ldots, d \\ P_j(\widetilde{\mathbf{y}}_j(k), \mathbf{z}_1(k), \ldots, \mathbf{z}_d(k)) = 0 \text{ for all } j = 1, \ldots, p \end{cases} \quad (13)$$

---

Notice that the algorithm $\mathrm{IPPartReal}$ depends on several other algorithms, such as $\mathrm{ApproxIdeal}$, $\mathrm{NoetherNorm}$ and $\mathrm{ComputeDepPoly}$. Each one of these algorithms can be implemented using techniques from commutative algebra, as we describe next.

The algorithm $\mathrm{ApproxIdeal}(D, M, N)$ computes an approximation of the Gröbner-basis of $I_N$ and proceeds as follows.

---

$\mathrm{ApproxIdeal}(D, M, N)$

---

1: For each $l = 0, \ldots, M$, let $I_{l,N}$ be the ideal generated by the polynomials $T_{kp+j} - \widetilde{\mathbf{y}}_j(k + l)$ for each $k = 0, \ldots, N$, and $j = 1, \ldots, p$.
2: Compute the Gröbner-basis $G_{N,M}$ of the ideal $I_{N,M} = \bigcap_{l=0,\ldots,M} I_{l,N}$ using the grlex ordering (see [23]). Return a Gröbner-basis of the ideal generated by those elements of $G_{N,M}$ that are of degree less than $D$.

---

The algorithm $\mathrm{NoetherNorm}(N, G_N)$ returns $d = \dim A_{\mathcal{Y},N}$ and a set of polynomials $Y_1, \ldots, Y_d$ in $\mathbb{R}[\mathcal{T}_N]$ such that the substitutions

$$\mathbf{z}_i = Y_i((\widetilde{\mathbf{y}}_1(k))_{k\in\mathbb{N}}, \ldots, (\widetilde{\mathbf{y}}_p(N + k))_{k\in\mathbb{N}}) \in \mathbb{R}^\infty \text{ for each } i = 1, \ldots, d$$

yield a *Noether Normalization* $\mathbf{z}_1, \ldots, \mathbf{z}_d$ of $\mathcal{A}_{\mathcal{Y},N}$. This algorithm is known to be computable from any finite basis $G_N$ of the ideal $I_N$, as can be seen from the proof of the Noether Normalization Theorem (see [22]).

The algorithm $\mathrm{DepPoly}(N, Y_1, \ldots, Y_d, Z, G_N)$ returns a nontrivial polynomial $Q$ in $d + 1$ variables such that $Q(Z, Y_1, \ldots, Y_d) \in I_N$ for polynomials $Z, Y_1, \ldots, Y_d \in \mathbb{R}[\mathcal{T}_N]$, provided that such a polynomial $Q$ exists. The algorithm proceeds as follows.

---

$\mathrm{ComputeDepPoly}(N, Y_1, \ldots, Y_d, Z, G_N)$

---

1: Introduce new variables $S_0, S_1, \ldots, S_d$ and define the ideal $J$ of the polynomial ring $\mathbb{R}[S_0, \ldots, S_d, T_1 \ldots T_{(N+1)p}]$ as the ideal generated by the elements of the Gröbner-basis of $G_N$ and the polynomials $S_0 - Z$ and $S_i - Y_i$, $i = 1, \ldots, d$.
2: Compute the Gröbner-basis $\hat{G}$ of the intersection $J \cap \mathbb{R}[S_0, S_1, \ldots, S_d]$, see [23] for an algorithm. Return an element $Q$ of $\hat{G}$.

---

From the Algebraic Sampling Theorem stated in [28] it follows that if $M$ and $D$ are large enough, then $\mathrm{ApproxIdeal}(D, M, N)$ returns a Gröbner-basis of $I_N$. Hence, we get the following.

**Lemma 1 (Partial realization)** *Assume alg-rank $H_\mathcal{Y} < +\infty$. Then, if $M$ and $D$ are large enough, then the IPS $\mathcal{P}_{\widetilde{\mathbf{y}}}$ returned by $\mathrm{IPPartReal}(M, D)$ is a realization of $\mathcal{Y} = (\widetilde{\mathbf{y}}(k))_{k\in\mathbb{N}}$, and the dimension of $\mathcal{P}_{\widetilde{\mathbf{y}}}$ is at most alg-rank $H_\mathcal{Y}$. If $\mathcal{P}_{\widetilde{\mathbf{y}}}$ satisfies Assumption 1, then $\dim \mathcal{P}_{\widetilde{\mathbf{y}}} =$ alg-rank $H_\mathcal{Y}$.*

The question that arises is how to check if the output of $\mathrm{IPPartReal}$ satisfies Assumption 1. To this end, we can assume without loss of generality that the polynomials from (13) are of the form $P_j = \sum_{r=0}^{K} Z_0^r P_{j,r}$ and $Q_i = \sum_{l=0}^{K} Z_0^l Q_{i,l}$ for some $K > 0$, where $P_{j,r}$ and $Q_{i,l}$ are polynomials in $Z_1, \ldots, Z_n$ for all $i = 1, \ldots, n$, and $j = 1, \ldots, p$. Assume that the Groebner-basis $G_N$ of $I_N$ is known. Denote by $\hat{Q}_{i,l}$ and $\hat{P}_{j,r}$ the polynomials in $\mathbb{R}[\mathcal{T}_N]$ obtained from $Q_{i,l}$ and $P_{j,r}$ by substituting $Y_m$ for $Z_m$, $m = 1, \ldots, n$. It is easy to see that the IPS $\mathcal{P}_p$ returned by $\mathrm{IPPartReal}$ satisfies Assumption 1 if the zero set in $\mathbb{R}^{(N+1)p}$ of the ideal $S_a$ generated by the set of polynomials $G_N \cup \{\hat{Q}_{i,l}, \hat{P}_{j,r} \mid i = 1, \ldots, n, j = 1, \ldots, p, l, r = 0, \ldots, K\}$ is empty. Checking emptiness of $S_a$ can be done using techniques from algebraic geometry, for example, by using procedures for deciding emptiness of semi-algebraic sets, see [26].

**Realization algorithm for SAHSs.** Assume that $\mathcal{P}_p$ is the IPS returned by the algorithm $\mathrm{IPPartReal}$. Assume that $\mathcal{P}_p$ satisfies Assumption 1 and it is a realization of $\mathcal{Y}$. Then, it follows that $SAH(\mathcal{P}_p)$ is an almost minimal SAH system realization of $\mathcal{Y}$ and $\dim SAH(\mathcal{P}_p) =$ alg-rank $H_{\mathcal{Y}}$. If the equations of the IPS $\mathcal{P}_p$ are known, then the equations of $SAH(\mathcal{P}_p)$ can be computed. However, in order to compute the initial state and the switching sequence of $SAH(\mathcal{P}_p)$ the knowledge of the states of $\mathcal{P}_p$ is required. Notice that $\mathrm{IPPartReal}$ also computes the state variables for time instances $k = 0, \ldots, M$.

**Realization algorithm for SASs.** We can proceed as follows. Use $\mathrm{IPPartReal}$ an IPS realization $\mathcal{P}_p$ of $\mathcal{Y}$. If $\mathcal{P}_p$ satisfies Assumption 1, we can use the procedure above to compute the equations and possibly the state of $\mathcal{H}_p = SAH(\mathcal{P}_p)$. It is easy to see that the knowledge of the equations of $\mathcal{H}_p$ allows us to compute the equations of $SA(\mathcal{H}_p)$. Unfortunately, the computation of the initial state of $SA(\mathcal{H}_p)$ is problematic, as it requires the knowledge of the whole infinite switching sequence. It follows that $SA(\mathcal{H}_p)$ is a realization of $\mathcal{Y}$ and $\dim SA(\mathcal{H}_p) =$ alg-rank $H_{\mathcal{Y}} + 1$.

## 5 Discussion and Future Work

We have presented necessary and an almost sufficient conditions for existence of a realization for implicit polynomial systems, semi-algebraic systems, and semi-algebraic hybrid systems, along with a characterization of minimality and a realization algorithm.

There are several potential directions for future research. To begin with, it would be desirable to find a sufficient condition for existence of a semi-algebraic realization. In addition, the relationship between minimality and such important properties as observability and reachability are not well-understood for semi-algebraic hybrid systems. Another potential research direction is to extend the results of the paper to systems with inputs, possibly stochastic. A third research direction could be to explore further the relationship between the approach presented in this paper and the works on identification using GPCA, see [19–21]. Extending the results of the paper to the continuous-time case represents a potential research direction as well. Investigating the computation complexity of the presented realization algorithm, remains a topic of future research.

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