Observability of Linear Hybrid Systems*

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Abstract. We analyze the observability of the continuous and discrete states of continuous-time linear hybrid systems. For the class of jump-linear systems, we derive necessary and sufficient conditions that the structural parameters of the model must satisfy in order for filtering and smoothing algorithms to operate correctly. Our conditions are simple rank tests that exploit the geometry of the observability subspaces. For linear hybrid systems, we derive weaker rank conditions that are sufficient to guarantee the uniqueness of the reconstruction of the state trajectory, even when the individual linear systems are unobservable.

1 Introduction

Observability refers to the study of the conditions under which it is possible to uniquely infer the state of a dynamical system from measurements of its output. For discrete-event systems, the definition of current-location observability was proposed in [12] as the property of being able to estimate the location of the system after a finite number of steps. A similar definition was given in [11] together with a polynomial test for observability, the so-called current-location tree, which depends on properties of the nodes of a finite state machine associated with the discrete-event system. For continuous systems with linear dynamics, it is well known that the observability problem can be reduced to that of analyzing the rank of the so-called *observability matrix*. This is the well known Popov-Belevic-Hautus rank test for linear systems [10]. For nonlinear systems with smooth dynamics, different definitions of observability have been proposed. We refer interested readers to [8] and references therein for a recent comparison of different definitions of observability. For hybrid systems, most of the previous work has concentrated on the areas of modeling, stability, controllability and verification (see previous workshop proceedings). Relatively little attention, however, has been devoted to the study of the observability of both the continuous and discrete states of a hybrid system.

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1.1 Prior work

To the best of our knowledge, the first attempt to characterize the observability of hybrid systems can be found in [15], where a definition of observability is proposed. [14] gives conditions for the observability of a particular class of linear time-varying systems where the system matrix is a linear combination of a basis with respect to time-varying coefficients. [6] addresses the observability and controllability of switched linear systems with known and periodic transitions. [13] gives a condition for the observability of switched linear systems in terms of the existence of a discrete state trajectory. [3] proposes the notion of incremental observability for piecewise affine systems. Such a notion requires the solution of a mixed-integer linear program in order to be tested. [16] derives necessary and sufficient conditions for the observability of discrete-time jump-linear systems. The conditions can be tested using simple simple rank tests on the structural parameters of the model. [5] derives different rank tests for the weak observability of jump-Markov linear systems. [9] gives observability conditions for stochastic linear hybrid systems in terms of the covariances of the outputs.

A problem related to observability that has been recently considered is the design of observers for linear hybrid systems. [1] considers the case in which the discrete state is known and proposes a Luenberger observer for the continuous state. [2] combines location observers with Luenberger observers to design a hybrid observer that identifies the discrete location in a finite number of steps and converges exponentially to the continuous state.

1.2 Contributions

In this paper we study the observability of a class of linear hybrid systems known as jump- (or switched-) linear systems, *i.e.,* systems whose evolution is determined by a collection of linear models with *continuous state* $x_t \in \mathbb{R}^n$ connected by switches among a number of *discrete states* $\lambda_t \in \{1, 2, \dots, N\}$. In Section 2 we introduce a notion of observability for jump-linear systems. We define the observability index ν of a jump-linear system and use it to derive rank conditions that the structural parameters of the model must satisfy in order for filtering and smoothing algorithms to operate correctly. We show that the state trajectory is observable if and only if the pairwise intersection of different observable subspaces is trivial. We also show that the switching times are observable if and only if the difference between any pair of observability matrices is nonsingular. The rank conditions we derive are simpler than their discrete-time counterparts [16] and can be thought of as an extension of the Popov-Belevic-Hautus rank test for linear systems. Our conditions only depend on the geometry of the observability subspaces, and therefore they are applicable also to the case of hybrid models where the switching mechanism depends on the continuous state. In Section 3 we derive weaker rank conditions that guarantee the observability of a linear hybrid system, even if the individual linear systems are unobservable. In this case, observability is gained by requiring that the output switches at least once in the given observability interval. Section 4 concludes the paper with a discussion about the role of the inputs in the observability of the discrete state.

2 Observability of jump-linear systems

We consider a class of continuous-time hybrid systems known as jump-linear systems, *i.e.,* systems whose evolution is determined by a collection of linear models with *continuous state* $x_t \in \mathbb{R}^n$ connected by switches of a number of *discrete states* $\lambda_t \in \{1, 2, \ldots, N\}$. The evolution of the continuous state x_t is described by the linear system

$$\dot{x}_t = A(\lambda_t)x_t \tag{1}$$

$$y_t = C(\lambda_t)x_t, \tag{2}$$

where $A(k) \in \mathbb{R}^{n\times n}$ and $C(k) \in \mathbb{R}^{p\times n}$, for $k \in \{1, 2, \ldots, N\}$. The evolution of the discrete state λ_t can be modeled, for instance, as an irreducible Markov chain governed by the transition map $\pi(t) \doteq P(\lambda_{t+1}|\lambda_t)$ or, as we do here, as a deterministic but unknown input that is piecewise constant, right-continuous and finite-valued[1]. Furthermore, we assume that the hybrid system admits no Zeno executions. More specifically, we assume that the switching times $\{t_i, i \geq 1\}$ are separated by at least $\tau > 0$; that is, we assume that $t_{i+1} - t_i \geq \tau > 0$. Having a minimum separation τ between consecutive switches is not a strong assumption to make, since τ can be arbitrarily small, as long as it is constant and positive.

Given a jump-linear system $\Sigma = \{A(k), C(k); k = 1, \ldots, N\}$, we focus our attention on how to infer the state of the system $\{x_t, \lambda_t\}$ from the output $\{y_t\}$. The simplest instance of this problem can be informally described as follows. Assume that we are given the model parameters $A(\cdot), C(\cdot)$ and that Σ evolves starting from an (unknown) initial condition (x_{t_0}, λ_{t_0}). Given the output $\{y_t\}$ in the interval $[t_0, t_0+T]$, is it possible to reconstruct the continuous state trajectory x_t and the discrete state trajectory λ_t uniquely?

If the sequence of discrete states $\lambda_{t_0}, \lambda_{t_1}, \ldots, \lambda_{t_0+T}$ is known, then the output of the system between two consecutive jumps can be written explicitly in terms of the model parameters $A(\cdot), C(\cdot)$, and the initial value of the continuous state x_{t_0} as

$$y_t = \begin{cases} C(\lambda_{t_0})e^{A(\lambda_{t_0})(t-t_0)}x_{t_0} & t \in [t_0, t_1) \\ C(\lambda_{t_1})e^{A(\lambda_{t_1})(t-t_1)}e^{A(\lambda_{t_0})(t_1-t_0)}x_{t_0} & t \in [t_1, t_2) \\ \vdots & \vdots \end{cases} \tag{3}$$

We thus propose the following notions of indistinguishability and observability.

[1] Most of the literature on hybrid systems restricts the switching mechanism of the discrete state to depend on the value of the continuous state. While this is generally sensible in the study of stability, it could be a significant restriction to impose in the context of filtering and identification. Therefore, our model is more general from an observability point of view, since it imposes no restriction on the mechanism that governs the transitions between discrete states. The conditions we derive are therefore sufficient for systems with state-dependent transitions.

Definition 1 (Indistinguishability). *We say that the states* $\{x_{t_0}, \lambda_t\}$ *and* $\{\bar{x}_{t_0}, \bar{\lambda}_t\}$ *are* **indistinguishable** *on the interval* $t \in [t_0, t_0+T]$ *if the corresponding outputs in free evolution* $\{y_t\}$ *and* $\{\bar{y}_t\}$ *are equal. We use* $\{x_{t_0}, \lambda_{t_0}, \dots, \lambda_{t_0+T}\}$ *instead of* $\{x_{t_0}, \lambda_t\}$ *to denote the state when the switching times are known. We denote the set of states which are indistinguishable from* $\{x_{t_0}, \lambda_t\}$ *as* $\mathcal{I}(x_{t_0}, \lambda_t)$*.*

Definition 2 (Observability). *We say that a state* $\{x_{t_0}, \lambda_t\}$ *is observable on* $t \in [t_0, t_0+T]$ *if* $\mathcal{I}(x_{t_0}, \lambda_t) = \{x_{t_0}, \lambda_t\}$*. When any admissible state is observable, we say that the model* Σ *is* **observable**.

Remark 1. Notice that we have defined observability in terms of the initial continuous state x_{t_0} and the discrete state evolution λ_t rather than in terms of the hybrid state evolution $\{x_t, \lambda_t\}$. This is because if $A(\cdot)$, x_{t_0} and λ_t are known, then x_t is automatically determined, similarly to (3).

2.1 Observability of the initial state

We first analyze the conditions under which we can determine x_{t_0} and $\lambda_t = \lambda_{t_0}$ for $t \in [t_0, t_1)$ uniquely, *i.e.,* before a switch occurs. We have that $\{x_{t_0}, \lambda_{t_0}\}$ is indistinguishable from $\{\bar{x}_{t_0}, \bar{\lambda}_{t_0}\}$ if and only if

$$C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_0)} x_{t_0} = C(\bar{\lambda}_{t_0}) e^{A(\bar{\lambda}_{t_0})(t-t_0)} \bar{x}_{t_0} \qquad \text{for } t \in [t_0, t_1). \tag{4}$$

After expanding both sides in Taylor series about t_0, the indistinguishability condition can be written as

$$y_{t_0}^{(k)} = C(\lambda_{t_0}) A(\lambda_{t_0})^k x_{t_0} = C(\bar{\lambda}_{t_0}) A(\bar{\lambda}_{t_0})^k \bar{x}_{t_0} \qquad \text{for } k \geq 0. \tag{5}$$

If we let $\mathcal{O}_\infty(\lambda_{t_0})$ and $\mathcal{O}_\infty(\bar{\lambda}_{t_0})$ be the infinite-dimensional extended observability matrices of the pairs $(A(\lambda_{t_0}), C(\lambda_{t_0}))$ and $(A(\bar{\lambda}_{t_0}), C(\bar{\lambda}_{t_0}))$, respectively, then the indistinguishability condition can be compactly written as

$$\mathcal{O}_\infty(\lambda_{t_0}) x_{t_0} = \mathcal{O}_\infty(\bar{\lambda}_{t_0}) \bar{x}_{t_0}. \tag{6}$$

Therefore, the initial state $\{x_{t_0}, \lambda_{t_0}\}$ is observable if and only if the rank condition $\operatorname{rank}([\mathcal{O}_\infty(\lambda_{t_0}) \;\; \mathcal{O}_\infty(\bar{\lambda}_{t_0})]) = 2n$ holds. It turns out that, as in the linear systems case, we can restrict our attention to finite-dimensional observability matrices, because the *extended joint observability matrix* $\mathcal{O}_\infty(k, k') \triangleq [\mathcal{O}_\infty(k) \; \mathcal{O}_\infty(k')]$ equals the extended observability matrix of the $2n$-dimensional system defined by

$$A(k, k') = \begin{bmatrix} A(k) & 0 \\ 0 & A(k') \end{bmatrix} \quad \text{and} \quad C(k, k') = \begin{bmatrix} C(k) \; C(k') \end{bmatrix}.$$

Hence, we define the *joint observability index* of systems k and k' as the minimum integer $\nu(k, k')$ such that the rank of the finite-dimensional joint observability matrix $\mathcal{O}_j(k, k') \triangleq [\mathcal{O}_j(k) \; \mathcal{O}_j(k')]$, where

$$\mathcal{O}_j(k) = [C(k)^T \; (C(k)A(k))^T \; \cdots \; (C(k)A(k)^{j-1})^T]^T, \tag{7}$$

stops growing. Thus, we can rephrase the indistinguishability condition in terms of the largest joint observability index $\nu \triangleq \max_{k\neq k'}\{\nu(k,k')\} \leq 2n$ as

$$\mathcal{Y}_\nu(t_0) \triangleq \begin{bmatrix} y_{t_0} \\ \dot{y}_{t_0} \\ \vdots \\ y_{t_0}^{(\nu-1)} \end{bmatrix} = \mathcal{O}_\nu(\lambda_{t_0})x_{t_0} = \mathcal{O}_\nu(\bar{\lambda}_{t_0})\bar{x}_{t_0}. \tag{8}$$

From this equation we derive the following condition on the observability of the initial state $\{x_{t_0}, \lambda_{t_0}\}$.

Lemma 1 (Observability of the initial state). *If $t_1 - t_0 \geq \tau > 0$, then the initial state $\{x_{t_0}, \lambda_{t_0}\}$ is observable if and only if for all $k \neq k' \in \{1, \ldots, N\}$ we have* $\operatorname{rank}([\mathcal{O}_\nu(k)\ \mathcal{O}_\nu(k')]) = 2n$*. Furthermore, the initial state is given by*

$$\lambda_{t_0} = \{k : \operatorname{rank}([\mathcal{O}_\nu(k)\ \ \mathcal{Y}_\nu(t_0)]) = n\} \quad \textit{and} \quad x_{t_0} = \mathcal{O}_\nu(\lambda_{t_0})^\dagger \mathcal{Y}_\nu(t_0), \tag{9}$$

where $M^\dagger = (M^T M)^{-1} M^T$.

We illustrate the applicability of Lemma 1 with the following example.

Example 1 (Two observable linear systems give an unobservable hybrid system). Consider the one-dimensional jump-linear system composed of the two linear systems

$$\begin{aligned} \dot{x} &= 0 \\ y &= c_1 x \end{aligned} \qquad \text{and} \qquad \begin{aligned} \dot{x} &= 0 \\ y &= c_2 x \ , \end{aligned} \tag{10}$$

where $c_1 \neq 0$, $c_2 \neq 0$ and $c_1 \neq c_2$. We observe that the initial state of each linear system is observable, but the initial state of the jump-linear system is not: One can set the initial condition of system 1 to x_0 and the initial condition of system 2 to $c_1 x_0 / c_2$ and obtain identical outputs. That is, states $\{x_0, 1\}$ and $\{c_1 x_0 / c_2, 2\}$ are indistinguishable. Notice that in this example the rank-$2n$ condition is violated, because $\operatorname{rank}([c_1\ c_2]) = 1 < 2$.

Remark 2 (Observability subspaces). Notice that the rank-$2n$ condition implies that each linear system $(A(k), C(k))$ must be observable, because it implies that $\operatorname{rank}(\mathcal{O}_\nu(k)) = n$ for all $k \in \{1, \ldots, N\}$. In addition, if we denote the range of $\mathcal{O}_\nu(k)$ as the observability subspace associated with linear system k, then the rank-$2n$ condition implies that the intersection of the observability subspaces of each pair of linear systems must be trivial. In fact, the set of unobservable states can be directly obtained from the intersection of the observability subspaces. One could therefore introduce a notion of distance between models using the angles between the observability subspaces, similarly to [4].

2.2 Observability of the first switching time

Lemma 1 provides conditions for the observability of the initial state $\{x_{t_0}, \lambda_{t_0}\}$. We are now interested in the observability of $\{x_{t_0}, \lambda_t\}$ for $t \in [t_0, t_1]$. Since λ_t is a piecewise constant function, we only need to concentrate on the conditions under which the first switching time, t_1, can be uniquely determined. The output of the jump-linear system is given by

$$y_t = \begin{cases} C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_0)} x_{t_0} & t \in [t_0, t_1) \\ C(\lambda_{t_1}) e^{A(\lambda_{t_1})(t-t_1)} e^{A(\lambda_{t_0})(t_1-t_0)} x_{t_0} & t \in [t_1, t_2) \end{cases} \tag{11}$$

and we want to determine if it is possible to also write the output as

$$y_t = \begin{cases} C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_0)} x_{t_0} & t \in [t_0, \bar{t}_1) \\ C(\lambda_{\bar{t}_1}) e^{A(\lambda_{\bar{t}_1})(t-\bar{t}_1)} e^{A(\lambda_{t_0})(\bar{t}_1-t_0)} x_{t_0} & t \in [\bar{t}_1, t_2) \end{cases} . \tag{12}$$

Without loss of generality, assume that $\bar{t}_1 > t_1$ and consider the output y_t in the interval $[t_1, \bar{t}_1)$. We observe that t_1 is indistinguishable if and only if for $t \in [t_1, \bar{t}_1)$

$$C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_1)} e^{A(\lambda_{t_0})(t_1-t_0)} x_{t_0} = C(\lambda_{t_1}) e^{A(\lambda_{t_1})(t-t_1)} e^{A(\lambda_{t_0})(t_1-t_0)} x_{t_0}. \tag{13}$$

After expanding both sides in Taylor series about t_1, the indistinguishability condition can be written as

$$y_{t_1}^{(k)} = C(\lambda_{t_0}) A(\lambda_{t_0})^k x_{t_1} = C(\lambda_{t_1}) A(\lambda_{t_1})^k x_{t_1} \qquad \text{for } k \geq 0. \tag{14}$$

As before, then the indistinguishability condition can be compactly written in terms of the extended observability matrices as

$$\mathcal{O}_\nu(\lambda_{t_0}) x_{t_1} = \mathcal{O}_\nu(\lambda_{t_1}) x_{t_1}. \tag{15}$$

Hence, t_1 is indistinguishable when the difference between the observability matrices $\mathcal{O}_\nu(\lambda_{t_0}) - \mathcal{O}_\nu(\lambda_{t_1})$ is singular. Since this could happen for any pair of observability matrices, in order for t_1 to be observable, we need to ensure that the difference of any pair of observability matrices is nonsingular. We therefore have the following Lemma on the observability of the first switching time.

Lemma 2 (Observability of the first switching time). *If $t_1 - t_0 \geq \tau > 0$, then the first switching time is observable if and only if for all $k \neq k' \in \{1, \ldots, N\}$ we have $\operatorname{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$. Furthermore, the first switching time can be recovered as the time instance at which the output y_t is not $\mathcal{C}^\infty$, i.e.,*

$$t_1 = \min\{t > t_0 : \mathcal{Y}_\nu(t^-) \neq \mathcal{Y}_\nu(t^+)\}. \tag{16}$$

Remark 3 (Continuous reset map). Notice that if the continuous reset is different from the identity map, then the switching times can be found by looking at the discontinuities of y_t directly, with no need for higher-order derivatives of y_t.

Remark 4 (Unobservable subspaces). Notice that if a continuous state x is unobservable for linear systems k and k', then $\mathcal{O}_\nu(k)x = \mathcal{O}_\nu(k')x = 0$, hence $(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k'))x = 0$. Therefore, the rank-$n$ condition $\mathrm{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$ implies that the intersection of the null-spaces of any pair of observability matrices, *i.e.,* the intersection of the unobservable subspaces, must be trivial. While this observation is irrelevant for the observability of a jump-linear system, because each linear system has to be observable (See Remark 2), it will be quite important for uniquely reconstructing the state trajectory of unobservable jump-linear systems, as we will discuss in Section 3.

2.3 Observability of jump-linear systems

Once x_{t_0}, λ_{t_0} and t_1 have been determined, we just repeat the process for the remaining jumps. The only difference is that x_{t_i}, $i \geq 1$, will be given. However, since λ_{t_0} is originally unknown, we still need to check the rank-$2n$ condition of Lemma 1 for any pair of extended observability matrices in order for x_{t_0} and λ_{t_0} to be uniquely recoverable. Therefore, since the rank-$2n$ condition $\mathrm{rank}([\mathcal{O}_\nu(k) \;\; \mathcal{O}_\nu(k')]) = 2n$ implies the rank-n condition $\mathrm{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$, we have the following theorem on the observability of jump-linear systems.

Theorem 1 (Observability of jump-linear systems). *If for all $i \geq 0$ we have $t_{i+1} - t_i \geq \tau > 0$, then $\{x_{t_0}, \lambda_t\}$ is observable on $t \in [t_0, t_0 + T]$ if and only if for all $k \neq k' \in \{1, \ldots, N\}$ we have $\mathrm{rank}([\mathcal{O}_\nu(k) \;\; \mathcal{O}_\nu(k')]) = 2n$. Furthermore, the state trajectory can be uniquely recovered as*

$$\lambda_{t_0} = \{k : \mathrm{rank}([\mathcal{O}_\nu(k) \;\; \mathcal{Y}_\nu(t_0)]) = n\}, \tag{17}$$

$$x_{t_0} = \mathcal{O}_\nu(\lambda_{t_0})^\dagger \mathcal{Y}_\nu(t_0), \tag{18}$$

$$t_i = \min\{t > t_{i-1} : \mathcal{Y}_\nu(t^-) \neq \mathcal{Y}_\nu(t^+)\}, \tag{19}$$

$$\lambda_{t_i} = \{k : \mathrm{rank}([\mathcal{O}_\nu(k) \;\; \mathcal{Y}_\nu(t_i)]) = n\}. \tag{20}$$

Remark 5 (Observability of discrete-time jump-linear systems). Notice that the rank conditions of Theorem 1 are simpler than their discrete-time counterparts. In discrete time, it is possible that a switch occurs at time t_i but its effect in the output appears some time steps after t_i. In that case, in order to guarantee observability, additional rank constraints need to be imposed, for example the $A(\cdot)$ matrices must be nonsingular and they cannot commute. We refer interested readers to [16] for more details about the discrete-time case.

Remark 6 (Observability of jump-linear systems in terms of observability operators). The rank constraints of Theorem 1 can also be expressed in terms of observability operators. For example, let $\mathcal{L}(k) : \mathbb{R}^n \to \mathcal{C}^\infty_{[0,\tau]}$ be defined as

$$x \mapsto y(t) = [\mathcal{L}(k)x](t) \triangleq C(k)\mathrm{e}^{A(k)t}x \quad \text{for} \quad t \in [0, \tau]. \tag{21}$$

Also let the adjoint observability operator $\mathcal{L}^*(k) : \mathcal{C}^\infty_{[0,\tau]} \to \mathbb{R}^n$ be defined as

$$\xi(\cdot) \mapsto x = \mathcal{L}^*(k)\xi \triangleq \int_0^\tau \mathrm{e}^{A(k)^T(\tau - s)} C(k)^T \xi(s)\, ds \quad \text{for} \quad \xi(\cdot) \in \mathcal{C}^\infty_{[0,\tau]}. \tag{22}$$

Then a linear hybrid system is observable if and only if for all $k \neq k' \in \{1, \ldots, N\}$ the range of the operator $\mathcal{L}(k) \times \mathcal{L}(k')$ is $2n$-dimensional. This implies that $Range(\mathcal{L}(k)) \cap Range(\mathcal{L}(k')) = \{0\}$ and that $\mathcal{L}(k) - \mathcal{L}(k')$ is injective. Then one can reconstruct the state trajectory by orthogonally projecting the output onto the range of these observability operators. More specifically, one can determine the initial discrete state λ_{t_0} by looking at k such that

$$y(t) - \left[\mathcal{L}(k)(\mathcal{L}^*(k)\mathcal{L}(k))^{-1}\mathcal{L}^*(k)y\right](t) = 0 \quad \forall t \in [0, \tau), \quad \tau \leq t_1.$$

Given λ_{t_0}, the initial continuous x_{t_0} can be determined as

$$x_{t_0} = (\mathcal{L}^*(\lambda_{t_0})\mathcal{L}(\lambda_{t_0}))^{-1}\mathcal{L}^*(\lambda_{t_0})y(t).$$

Similarly, the first switching time, t_1, can be determined as the first time instant $t > t_0$ such that

$$y(t) - \left[\mathcal{L}(\lambda_{t_0})(\mathcal{L}^*(\lambda_{t_0})\mathcal{L}(\lambda_{t_0}))^{-1}\mathcal{L}^*(\lambda_{t_0})y\right](t) \neq 0.$$

The same argument applies for the subsequent discrete states and switching times.

Remark 7. For ease of exposition and in order to make the connection with the discrete-time case, we have chosen to state our results in terms of derivatives of the output. Nevertheless, when it comes to doing computations and quantifying errors, working with grammians may turn out to be more convenient. Just to mention one point, in practice one needs to quantify "how far" two linear systems are and how this affects the estimation of the initial state and of the discrete sequence in the presence of "noise". If one considers, for instance, L^2 distances in the output spaces $d_{[0,\tau]}(k) = \int_0^\tau \|y(t) - \left[\mathcal{L}(k)(\mathcal{L}^*(k)\mathcal{L}(k))^{-1}\mathcal{L}^*(k)y\right](t)\|^2\, dt$ then a natural way to measure the distance between two systems is by looking at the subspace angles between observability subspaces, as suggested in [4]. In fact, assume that $y(t)$ has been generated by system 1 and we measure $d_{[0,\tau]}(2)$ then it holds that $d_{[0,\tau]}(2) \geq \|y\|^2 \sin^2(\theta_{min})$ where θ_{min} is the smallest canonical angle between $Range(\mathcal{L}(1))$ and $Range(\mathcal{L}(2))$. Investigating these issues will be the subject of future research.

Remark 8 (Role of the input in the observability of the discrete state). The notion of observability we have proposed does not depend on the input. This is consistent with the standard theory for linear systems. However, identifying the discrete state of a jump-linear system is equivalent to a system identification problem, where the class of possible models is restricted to a finite set. Unlike observability, identifiability – even for linear systems – does depend on the input[2], and therefore the input ought to play a role in the identification of linear hybrid systems. We discuss this further in Section 4.

[2] This fact was pointed out to us by Prof. Claire Tomlin (personal communication).

3 Observability of linear hybrid systems

Theorem 1 gives *necessary and sufficient* conditions for the observability of a class of linear hybrid systems known as jump-linear systems. Since the theorem imposes no restriction on the mechanism that governs the transitions between discrete states, the conditions of Theorem 1 remain *sufficient* for other classes of linear hybrid systems in which the switching mechanism depends on the value of the continuous state, e.g. piecewise affine systems, as long as there is a minimum separation $\tau > 0$ between consecutive switches. This is because, given a linear hybrid system $\mathcal{H}$, one can always associate with it a jump-linear systems Σ that abstracts the discrete behavior as well as the interaction between discrete and continuous states defined by the guards and invariants. Then, if the jump-linear system Σ is observable, so is the linear hybrid system $\mathcal{H}$.

However, the conditions of Theorem 1 are not *necessary* for the observability of a linear hybrid system. In fact, there are cases in which the associated jump-linear system is itself unobservable (in the sense of Definition 2), yet it is possible to uniquely reconstruct the state trajectory from a particular output[3]. Intuitively, this happens when the linear hybrid system switches from an unobservable state for system k to an observable one for system k', as we illustrate in the following example. (See [3] for additional examples in discrete-time).

Example 2 (Unique reconstruction from two unobservable linear systems). Consider a two-dimensional linear hybrid system composed of the two linear systems

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x & \qquad & \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x \\ & & \text{and} & \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x & & y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \,. \end{aligned} \tag{23}$$

Let $t_0 = 0$, $T = 2$, $x_0 = [0, 1]^T$ and assume that there is a single switch from system 1 to system 2 at time $t_1 = 1$. Then $\nu = 2$,

$$x_t = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}, \quad y_t = \begin{cases} 0 & t \in [0,1) \\ e^{2t} & t \in [1,2) \end{cases}, \quad \mathcal{O}_2(1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{O}_2(2) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

In this example, both linear systems are unobservable and the initial condition lies in the unobservable subspace of system 1. Also notice that the rank-$2n$ condition is violated, since $\text{rank}([\mathcal{O}_2(1) \;\; \mathcal{O}_2(2)]) = 2 < 4$, thus Lemma 1 does not apply. However, the first switching time can be uniquely recovered, because the rank-n condition of Lemma 2 holds since $\text{rank}(\mathcal{O}_2(2) - \mathcal{O}_2(1)) = 2$. In fact, y_t is discontinuous at $t_1 = 1$. Furthermore, one can uniquely reconstruct x_{t_0}, λ_{t_0} and λ_{t_1}, because the unobservable subspace of system 1 is observable for system 2 and vice-versa. We make this more precise in the rest of this Section.

[3] Notice that this is impossible for linear systems. A linear system is either observable, in which case one can uniquely reconstruct the state for any given nonzero output, or it is unobservable, in which case for any output there are always infinitely many possible state trajectories generating it.

Following Example 2, in this section we derive weaker sufficient conditions under which one can uniquely reconstruct the state trajectory of a linear hybrid system given a particular output. We show how this can be done despite the individual linear systems being unobservable or the output of the system being zero during a switching interval. We start by assuming that we know the number and location of the switching times in the interval $[t_0, t_0+T]$. According to our discussion in the previous section, this is equivalent to assuming that for all $k \neq k' \in \{1, \ldots, N\}$ we have $\text{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$. Notice that this rank-$n$ condition does not require the individual linear systems to be observable. Now, from the indistinguishability condition

$$\mathcal{Y}_\nu(t_0) = \mathcal{O}_\nu(\lambda_{t_0})x_{t_0} = \mathcal{O}_\nu(\bar{\lambda}_{t_0})\bar{x}_{t_0}, \tag{24}$$

we have that the initial discrete state is indistinguishable whenever the intersection of any pair of observability subspaces is nontrivial, which happens if

$$\text{rank}([\mathcal{O}_\nu(k)\ \mathcal{O}_\nu(k')]) < \text{rank}(\mathcal{O}_\nu(k)) + \text{rank}(\mathcal{O}_\nu(k')). \tag{25}$$

We thus have the following:

1. If $\mathcal{Y}_\nu(t_0) \neq 0$, then the discrete state λ_{t_0} can be uniquely recovered provided that the intersection of any pair of observability subspaces is trivial. That is, for all $k \neq k' \in \{1, \ldots, N\}$ we must have

$$\text{rank}([\mathcal{O}_\nu(k)\ \mathcal{O}_\nu(k')]) = \text{rank}(\mathcal{O}_\nu(k)) + \text{rank}(\mathcal{O}_\nu(k')). \tag{26}$$

In this case we have

$$\lambda_{t_0} = \{k : \text{rank}([\mathcal{O}_\nu(k)\ \ \mathcal{Y}_\nu(t_0)]) = \text{rank}(\mathcal{O}_\nu(k))\}. \tag{27}$$

Notice that we do not need $\mathcal{O}_\nu(k)$ to be full rank, hence the rank-$2n$ condition may be violated here.

2. If $\mathcal{Y}_\nu(t_0) \neq 0$ and $t_1 > t_0 + T$, *i.e.,* if there is no switch during the observability window, then the continuous state can be uniquely recovered if and only if each linear system is observable, *i.e.,* if for all $k \in \{1, \ldots, N\}$ we have $\text{rank}(\mathcal{O}_\nu(k)) = n$. Let λ_{t_0} be defined as in (27), then we have

$$x_{t_0} = \mathcal{O}_\nu(\lambda_{t_0})^\dagger \mathcal{Y}_\nu(t_0). \tag{28}$$

This means that, if there is no switch, then we *do* need every system to be observable, hence the rank-$2n$ condition has to be in effect.

3. If $\mathcal{Y}_\nu(t_0) \neq 0$ and $t_1 < t_0 + T$, *i.e.,* if at least one switch occurs during the observability window, then the continuous state may *not* be uniquely recovered from the output in the interval $[t_0, t_1)$, but one may still be able to uniquely recover it from the output on the whole interval $[t_0, t_0+T]$. Loosely speaking, we need to find a condition such that the part of x_{t_0} that is not observable on $[t_0, t_1)$ becomes observable on $[t_1, t_0+T]$. For example, imagine that there is only one switch at time t_1. Then we have that

$$\begin{bmatrix} \mathcal{O}_\nu(\lambda_{t_0}) \\ \mathcal{O}_\nu(\lambda_{t_1})e^{A(\lambda_{t_0})(t_1-t_0)} \end{bmatrix} x_{t_0} = \begin{bmatrix} \mathcal{Y}_\nu(t_0) \\ \mathcal{Y}_\nu(t_1) \end{bmatrix}. \tag{29}$$

Therefore, in order to determine x_{t_0} uniquely we need

$$\operatorname{rank}\begin{bmatrix}\mathcal{O}_\nu(\lambda_{t_0})\\ \mathcal{O}_\nu(\lambda_{t_1})e^{A(\lambda_{t_0})(t_1-t_0)}\end{bmatrix} = n. \tag{30}$$

This rank condition is trivially satisfied, because the null-space of $\mathcal{O}_\nu(\lambda_{t_0})$ is $e^{A(\lambda_{t_0})}$-invariant and we have assumed that $\operatorname{rank}(\mathcal{O}(\lambda_{t_1}) - \mathcal{O}(\lambda_{t_0})) = n$ in order for t_1 to be observable (See Remark 4).
More generally, if there are j switches, $t_1, t_2, \ldots t_j$, on the interval $[t_0, t_0+T]$, $\mathcal{Y}_\nu(t_i) \neq 0$ for $i = 0, 1, \ldots, j$, and the corresponding sequence of discrete states $\lambda_{t_0}, \lambda_{t_1}, \ldots, \lambda_{t_j}$ can be uniquely recovered similarly to (27), then the initial continuous state x_{t_0} can be uniquely recovered from

$$\begin{bmatrix}\mathcal{O}_\nu(\lambda_{t_0})\\ \mathcal{O}_\nu(\lambda_{t_1})e^{A(\lambda_{t_0})(t_1-t_0)}\\ \vdots\\ \mathcal{O}_\nu(\lambda_{t_j})e^{A(\lambda_{t_{j-1}})(t_j-t_{j-1})}\cdots e^{A(\lambda_{t_0})(t_1-t_0)}\end{bmatrix} x_{t_0} = \begin{bmatrix}\mathcal{Y}_\nu(t_0)\\ \mathcal{Y}_\nu(t_1)\\ \vdots\\ \mathcal{Y}_\nu(t_j)\end{bmatrix}. \tag{31}$$

Notice again that the matrix on the left is full rank thanks to the rank-n condition $\operatorname{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$.

4. If $\mathcal{Y}_\nu(t_0) = 0$, then we cannot compute λ_{t_0} from (27). However, the rank constraint in (30) guarantees that $\mathcal{Y}_\nu(t_1) = \mathcal{O}_\nu(\lambda_{t_1})e^{A(\lambda_{t_0})(t_1-t_0)}x_{t_0} \neq 0$. Therefore, we can solve for λ_{t_1} uniquely similarly to (27). Given λ_{t_1}, the rank-n condition $\operatorname{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$ guarantees that λ_{t_0} can be uniquely determined as

$$\lambda_{t_0} = \{k : \operatorname{rank}\begin{bmatrix}\mathcal{O}_\nu(k) & 0\\ \mathcal{O}_\nu(\lambda_{t_1}) & \mathcal{Y}_\nu(t_1)\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathcal{O}_\nu(k)\\ \mathcal{O}_\nu(\lambda_{t_1})\end{bmatrix}\}. \tag{32}$$

More generally, whenever the output is zero in an interval $[t_i, t_{i+1})$, *i.e.*, whenever $\mathcal{Y}_\nu(t_i) = 0$, we must have that $\mathcal{Y}_\nu(t_{i+1}) \neq 0$ from which we can uniquely recover $\lambda_{t_{i+1}}$ as in (27). Given $\lambda_{t_{i+1}}$ one can uniquely determine λ_{t_i} as in (32). Then we are back into the situation of step 3 in which the discrete sequence is known, hence x_{t_0} can be uniquely recovered from (31).

We summarize our discussion in the following Theorem.

Theorem 2 (Observability of linear hybrid systems). *Consider a linear hybrid system $\mathcal{H}$ such that the switching times satisfy $t_{i+1} - t_i \geq \tau \ \forall i \geq 0$, and let $\Sigma = \{A(k), C(k); k = 1, \ldots, N\}$ be the associated jump-linear system. We have the following.*

1. ***Observability of the switching times**: If the difference between any pair of observability matrices is nonsingular; that is, if*

$$\textit{for all } k \neq k' \in \{1, \ldots, N\} \textit{ we have } \operatorname{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n, \tag{33}$$

then the switching times can be uniquely recovered as the time instances at which the output y_t is not $\mathcal{C}^\infty$, that is

$$t_i = \min\{t > t_{i-1} : \mathcal{Y}_\nu(t^-) \neq \mathcal{Y}_\nu(t^+)\}. \tag{34}$$

We denote by j the total number of switches in the interval $[t_0, t_0 + T]$.

2. ***Observability of the discrete state trajectory**: If in addition the intersection of the observability subspaces of any pair of observability matrices is trivial, that is if for all $k \neq k' \in \{1, \dots, N\}$ we have*

$$\operatorname{rank}([\mathcal{O}_\nu(k)\ \mathcal{O}_\nu(k')]) = \operatorname{rank}(\mathcal{O}_\nu(k)) + \operatorname{rank}(\mathcal{O}_\nu(k')), \tag{35}$$

then the discrete state trajectory can be uniquely recovered as follows:

(a) *For the switching times t_i such that $\mathcal{Y}_\nu(t_i) \neq 0$, obtain the discrete state similarly to (27) as*

$$\lambda_{t_i} = \{k : \operatorname{rank}([\mathcal{O}_\nu(k)\ \ \mathcal{Y}_\nu(t_i)]) = \operatorname{rank}(\mathcal{O}_\nu(k))\}. \tag{36}$$

(b) *For the switching times t_i such that $\mathcal{Y}_\nu(t_i) = 0$,*
   - *Compute $\lambda_{t_{i+1}}$ similarly to (36) as*

$$\lambda_{t_{i+1}} = \{k : \operatorname{rank}([\mathcal{O}_\nu(k)\ \ \mathcal{Y}_\nu(t_{i+1})]) = \operatorname{rank}(\mathcal{O}_\nu(k))\}. \tag{37}$$

   - *Compute λ_{t_i} similarly to (32) as*

$$\lambda_{t_i} = \{k : \operatorname{rank}\begin{bmatrix} \mathcal{O}_\nu(k) & 0 \\ \mathcal{O}_\nu(\lambda_{t_{i+1}}) & \mathcal{Y}_\nu(t_{i+1}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \mathcal{O}_\nu(k) \\ \mathcal{O}_\nu(\lambda_{t_{i+1}}) \end{bmatrix}\}. \tag{38}$$

3. ***Observability of the initial continuous state**: Under the conditions stated before, the initial value of the continuous state can be uniquely recovered as*

$$x_{t_0} = \begin{bmatrix} \mathcal{O}_\nu(\lambda_{t_0}) \\ \mathcal{O}_\nu(\lambda_{t_1}) e^{A(\lambda_{t_0})(t_1 - t_0)} \\ \vdots \\ \mathcal{O}_\nu(\lambda_{t_j}) e^{A(\lambda_{t_{j-1}})(t_j - t_{j-1})} \cdots e^{A(\lambda_{t_0})(t_1 - t_0)} \end{bmatrix}^\dagger \begin{bmatrix} \mathcal{Y}_\nu(t_0) \\ \mathcal{Y}_\nu(t_1) \\ \vdots \\ \mathcal{Y}_\nu(t_j) \end{bmatrix}. \tag{39}$$

Example 3 (Unique reconstruction of the state of a linear hybrid system composed of two unobservable linear systems). Consider the two-dimensional linear hybrid system of Example 2, where $t_0 = 0$, $t_1 = 1$, $T = 2$, $x_0 = [0, 1]^T$ and

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} \qquad \begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned} \qquad x_t = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} \qquad y_t = \begin{cases} 0 & t \in [0, 1) \\ e^{2t} & t \in [1, 2) \end{cases}. \tag{40}$$

In this example we have $\nu = 2$,

$$\mathcal{O}_2(1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{O}_2(2) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathcal{Y}_2(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathcal{Y}_2(1) = \begin{bmatrix} e^2 \\ 2e^2 \end{bmatrix}. \tag{41}$$

Therefore, both linear systems are unobservable and the initial condition lies in the null-space of $\mathcal{O}_2(1)$. Also notice that the rank-$2n$ condition is violated, since $\operatorname{rank}([\mathcal{O}_2(1)\ \ \mathcal{O}_2(2)]) = 2 < 4$, thus Lemma 1 does not apply. However, we have $\operatorname{rank}(\mathcal{O}_2(1) - \mathcal{O}_2(2)) = 2$, thus t_1 can be uniquely recovered, because y_t is discontinuous at $t_1 = 1$. Also $\operatorname{rank}([\mathcal{O}_2(1)\ \ \mathcal{O}_2(2)]) = \operatorname{rank}(\mathcal{O}_2(1)) + \operatorname{rank}(\mathcal{O}_2(2)) = 1+1 = 2$, thus $\lambda_{t_1} = 2$ can be uniquely recovered, because $\operatorname{rank}([\mathcal{O}_2(1)\ \mathcal{Y}_2(1)]) = 2 \neq 1$ while $\operatorname{rank}([\mathcal{O}_2(2)\ \mathcal{Y}_2(1)]) = 1 = 1$. Given t_1 and λ_{t_1}, one can estimate $\lambda_{t_0} = 1$ uniquely from (38), and $x_{t_0} = [0, 1]^T$ uniquely from (39).

4 Conclusions, discussion and open issues

We have presented an analysis of the observability of the continuous and discrete states of linear hybrid systems. For jump-linear systems, we demonstrated that under mild assumptions one can derive necessary and sufficient conditions that the structural parameters of the model must satisfy in order to guarantee the observability of the system. Our characterization is simple and intuitive and sheds light on the geometry of the observability subspaces generated by the output of a jump-linear system. For linear hybrid systems, we derived weaker rank conditions that guarantee the uniqueness of the reconstruction of the state trajectory, even if the individual linear systems are unobservable. In this case, observability is gained by requiring that the given output switches at least once in the observability interval. Although the conditions we have derived are sufficient for the observability of linear hybrid systems in which the switching mechanism depends on the value of the continuous state, e.g. piecewise affine systems, in the near future we expect to obtain weaker conditions that are also necessary.

An important issue that we did not addressed is concerned with characterizing the set of observationally equivalent models. In linear systems theory, this is done elegantly by the Kalman decomposition, which partitions the state space into orthogonal subspaces. Future work will address a characterization of this set for linear hybrid models.

Other aspects which remain to be investigated are the effect of measured inputs on the observability. The analysis we have carried out in the first part of the paper is limited to the case where the system evolves starting from some initial state with no driving input. The conditions we have derived, therefore, involve only the matrices $A(\cdot)$ and $C(\cdot)$. As we have anticipated in Remark 8, this is only part of the story. In fact, estimating the discrete state can be interpreted as the identification of a model within a finite number of possible models, and from a finite set of data. Therefore, unfortunately, one cannot use asymptotic results since, by assumption, switches occur in finite time. Instead, conditions on the input, such as persistence of excitation, that play a crucial role in system identification, will likely play an important role in the observability of hybrid systems too. For instance, let us consider a simple example where $N = 2$ and the two linear systems have *identical* $A(\cdot)$ and $C(\cdot)$ matrices, but different input-to-state coefficients (say for instance that $B(1) = 2B(2)$). Assume that the system is excited with white Gaussian noise. It is easy to prove that as the discrete state jumps from system 1 to system 2, the variance of the state increases and so does the variance of the output. Therefore, one should be able to detect a jump even though the $A(1) = A(2)$ and $C(1) = C(2)$, and hence the two models are indistinguishable according to our definition. This very simple example should caution the reader that a sensible definition of observability ought to also involve the input matrices (B, D). One may be tempted to give observability conditions in terms of the covariances of the outputs, but again this does not appear promising for at least two reasons. First, one can never compute good approximations of stationary covariances from finite sequences of data (in between switches). Second, well-known results [7] show that the output covariance of a jump-Markov linear system can be realized with a finite-dimensional ARMA model, and therefore covariance data are not sufficient to guarantee identifiability. The quest for different statistics (for instance, higher-order functions of the data) is worth investigating for this problem.

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