Attribute fusion in a latent process model for time series of graphs
Minh Tang, Youngser Park, Nam H. Lee, and Carey E. Priebe

Appendix
Proofs of some stated results

Corollary 2: The maximizer of $\mu_\lambda$ also maximizes

$$\mu_\lambda^2 = \frac{\lambda^T \xi^T \lambda}{\lambda^T \xi \lambda}$$

Because $\xi$ is positive definite, there exists a positive definite matrix $\xi^{1/2}$ such that $\xi^{1/2} \xi^{1/2} = \xi$. Letting $\nu = \xi^{1/2} \lambda$, the above expression can be rewritten as

$$\mu_\lambda^2 = \frac{\nu^T \xi^{-1/2} \xi^{-1/2} \nu}{\nu^T \nu}$$

The claim then follows directly from the Rayleigh-Ritz theorem for Hermitian matrices. □

Lemma 3: $\tau_2(G)$ is a U-statistics with kernel function $h(Y_1, Y_2, Y_3) = Y_1 Y_2 Y_3$. By the theory of U-statistics, we know that

$$\tau_2(G) - \mathbb{E}[\tau_2^*(G)] \sim \mathcal{N}(0, \text{Var}[\tau_2^*(G)])$$

provided that $\text{Var}[\tau_2(G) - \tau_2^*(G)] = o(\text{Var}[\tau_2^*(G)])$.

By the independent edge assumption, we have

$$\mathbb{E}[h(Y_1, Y_2, Y_3)] = \mathbb{E}[Y_1] \mathbb{E}[Y_2] \mathbb{E}[Y_3]$$

Thus, for $t < t^*$, we have

$$\mathbb{E}[\tau_2^*(G(t))] = \binom{n}{3} \langle \lambda, \pi_0 \rangle$$

and

$$\text{Var}[\tau_2^*(G(t))] = \sum_{u \neq v} \text{Var}[Y_{uv}]$$

We can thus decompose $\text{Var}[\tau_2^*(G(t))]$ as

$$\text{Var}[\tau_2^*(G(t))] = S_1^2 \sum_{u \neq v} \text{Var}[Y_{uv}] + S_2^2 \sum_{u \neq v} \text{Var}[Y_{uv}]$$

Now, for $\{u, v\} \in \mathcal{S}_1$, we have

$$S_1 Y_{uv} = \sum_{w \neq u, v} \mathbb{E}[h(Y_{uw}, Y_{uw}, Y_{uw})] Y_{uw}$$

The above expression is reasoned as follows. If $w \in [m]$, then $\mathbb{E}[Y_{uw}] = \mathbb{E}[Y_{uw}] = \langle \lambda, \pi_1 \rangle$ and there are $m - 2$ possible choices for $w \in [m]$ different from $u$ and $v$. If $w \in [n] \setminus [m]$, then $\mathbb{E}[Y_{uv}] = \mathbb{E}[Y_{uw}] = \langle \lambda, \pi_1 \rangle$ and there are $n - m$ possible choices for $w$. Analogous reasoning gives the expressions for $S_2$ and $S_3$ in the statement of the lemma.

We also have

$$\text{Var}[Y_{uv}] = \begin{cases} \langle \lambda, \eta_0 \rangle & \text{if } \{u, v\} \in \mathcal{S}_1, \\ \langle \lambda, \eta_1 \rangle & \text{if } \{u, v\} \in \mathcal{S}_2, \\ \langle \lambda, \eta_2 \rangle & \text{if } \{u, v\} \in \mathcal{S}_3 \end{cases}$$

and thus

$$\text{Var}[\tau_2^*(G(t))] = \binom{m}{3} \langle \lambda, \eta_0 \rangle S_1^2 + (m - n - m) \langle \lambda, \eta_1 \rangle S_2^2 + (m - m) \langle \lambda, \eta_2 \rangle S_3^2$$

as desired. To complete the proof one must show that $\text{Var}[\tau_2(G) - \tau_2^*(G)] = o(\text{Var}[\tau_2^*(G)])$ and this follows directly from the argument in [1] or [2]. □

Proposition 5: Let $v \in V(t)$ and denote by $d_\lambda(v; t)$ the (fused) degree of vertex $v$, i.e.,

$$d_\lambda(v; t) = \sum_{w \in N(v)} \langle \lambda, \Gamma_{vw} \rangle$$

For $t < t^*$, each of the $\Gamma_{vw}$ is a multinomial trial with probability vector $\pi_0$. The following statements are made as $n \to \infty$ for fixed $K$. By the central limit theorem, we have

$$d_\lambda(v; t) - (n - 1) \langle \lambda, \pi_{00} \rangle \to \mathcal{N}(0, 1)$$

We can thus consider the degree sequence of $G(t)$ for $t < t^*$ as a sequence of dependent normally distributed random variables. By an argument analogous to the argument for Erdős-Renyi random graphs in [3, §III.1] we can show that
the dependency among the \( \{d_{j}(v; t)\}_{v \in \mathcal{V}(t)} \) can be ignored. Another way of doing this is to note that the covariance between \( \mathcal{X}_{u} \) and \( \mathcal{X}_{v} \), where \( \mathcal{X}_{u} \) and \( \mathcal{X}_{v} \) are the ratio in Eq. (8) for vertices \( u \) and \( v \), is given by

\[
r = \text{Cov}(\mathcal{X}_{u}, \mathcal{X}_{v}) = \frac{3(\lambda, \tau_{00})}{\sqrt{(n-1)(\lambda, \eta_{00})}}.
\]

Because \( r \log n \to 0 \) as \( n \to \infty \), the sample maximum of the \( \mathcal{X}_{u} \) converges to the sample maximum of a sequence of independent \( \mathcal{N}(0, 1) \) random variables, \( d_{j}(v; t) \), can thus be considered as a sequence of independent random variables from a normal distribution. It is well known that the sample maximum of standard normal random variables converges weakly to a Gumbel distribution [4, §2.3]. It is, however, not clear whether the convergence of \( \Delta_{j}(t) \) to a Gumbel distribution continues to hold under the composition of weak convergence as outlined above. We avoid this problem by showing directly that

\[
\mathbb{P}\left( \frac{\Delta_{j}(t)-(n-1)(\lambda_{00}, \tau_{00})}{\sqrt{n-1}(\lambda, \eta_{00})} \leq a_{n} + b_{n}x \right) \to e^{-e^{-x}}.
\]

Let \( \zeta_{v} = \frac{d_{j}(v; t)-(n-1)(\lambda_{00}, \tau_{00})}{\sqrt{(n-1)(\lambda, \eta_{00})}} \) and \( F_{\zeta}(u) = \mathbb{P}(\zeta_{v} \leq u) \). If \( n \to \infty \) and \( u = O(\sqrt{\log n}) \), we have the following moderate deviations result [5], [6], Theorem 2, §XVI.7:

\[
\frac{1 - F_{\zeta}(u)}{1 - \Phi(u)} = \left[ 1 + \left( C \frac{u}{\sqrt{n}} \right) + O\left( \frac{u}{n} \right) \right]
\]

for some constant \( C \). Letting \( u_{n} = a_{n} + b_{n}x \) in Eq. (11), we have

\[
F_{\zeta}(u_{n}) = 1 - (1 - \Phi(u_{n}))(1 + C \frac{u_{n}}{\sqrt{n}} + O\left( \frac{u_{n}}{n} \right)) = \Phi(u_{n}) + (1 - \Phi(u_{n}))(C \frac{u_{n}}{\sqrt{n}} + O\left( \frac{u_{n}}{n} \right)) = \Phi(u_{n}) + O\left( \frac{u_{n}^{2}}{n} \right) + O\left( \frac{u_{n}}{n^{3/2}} \right)
\]

for some sufficiently small \( \delta > 0 \). We therefore have

\[
\mathbb{P}(\max_{v \in [n]} \zeta_{v} \leq u_{n}) = \left( F_{\zeta}(u_{n}) \right)^{n} = \left( \Phi(u_{n}) + O\left( \frac{u_{n}}{n^{3/2}} \right) \right)^{n} \to e^{-e^{-x}}.
\]

Eq. (10) is established and we obtain the limiting Gumbel distribution for \( \Delta_{j}(t) \) for \( t < t^{*} \).

The case when \( t = t^{*} \) can derive in a similar manner. We first show that if \( m = \Omega (\sqrt{n \log n}) \) then \( \Delta_{j}(v; t^{*}) \to \max_{v \in [m]} d_{j}(v; t^{*}) \) \( \mathcal{N}(0, 1) \). It then follows, similar to our previous reasoning for the case where \( t < t^{*} \), that \( \max_{v \in [m]} d_{j}(v; t^{*}) \to \mathcal{N}(0, 1) \). Then we show, again by the central limit theorem, that for \( v \in [m] \), \( \frac{d_{j}(v; t^{*}) - \mu_{j}}{\sigma_{j}} \to \mathcal{N}(0, 1) \). Then it follows, similar to our previous reasoning for the case where \( t < t^{*} \), that \( \max_{v \in [m]} \frac{d_{j}(v; t^{*}) - \mu_{j}}{\sigma_{j}} \to \mathcal{G}(\mu_{m}, \sigma_{m}) \) and we obtain the limiting Gumbel distribution for \( \Delta_{j}(t) \) for \( t = t^{*} \).

**Theorem 6:** Let \( X \sim \mathcal{G}(\alpha, \beta) \). We consider the normalization \( \frac{X - \mu}{\sigma} \). We have

\[
\mathbb{P}\left[ \frac{X - \mu}{\sigma} \leq z \right] = \mathbb{P}[X \leq z\sigma + \mu] = e^{-e^{-(z\sigma + \mu)\beta}} = e^{-e^{-(z - (\sigma + \mu)/\sigma)\beta}}.
\]

Thus, \( \frac{X - \mu}{\sigma} \to \mathcal{G}(\frac{\mu - \mu}{\sigma}, \beta) \). Because the sample mean and the sample variance are consistent estimators, the claim follows after an application of Slutsky’s theorem.

**Lemma 7:** Let \( \phi_{j}(v; t) = \psi_{j}(v; t) - d_{j}(v; t) \) be the (fused) locality statistics for vertex \( v \) at time \( t \) not including the (fused) degree of \( v \), i.e.,

\[
\phi_{j}(v; t) = \sum_{u \in \mathcal{N}(v), u \neq v} \langle \lambda, \Gamma_{uw} \rangle.
\]

The following statements are conditional on \( |\mathcal{N}(v)| = l \). First of all, we have

\[
\phi_{j}(v; t) = \sum_{k=1}^{K} \lambda_{k} z_{k}
\]

where the \( (z_{1}, \ldots, z_{K}) \) are distributed

\[
(z_{1}, z_{2}, \ldots, z_{K}) \sim \text{multinomial} \left( \left( \begin{array}{c} l \\ z \end{array} \right), \tau_{00} \right).
\]

By the central limit theorem, we have

\[
\frac{\phi_{j}(v; t) - \left( \begin{array}{c} l \\ \lambda \end{array} \right) \langle \lambda, \tau_{00} \rangle}{\sqrt{\left( \begin{array}{c} l \\ \lambda \end{array} \right) \langle \lambda, \eta_{00} \lambda \rangle}} \to \mathcal{N}(0, 1).
\]

Let \( \lambda^{(2)} \) be the element-wise square of \( \lambda \). Define \( C_{00} \) and \( p_{00} \) to be

\[
C_{00} = \frac{\langle \lambda^{(2)} \rangle \langle \lambda, \tau_{00} \rangle}{\langle \lambda, \eta_{00} \lambda \rangle}, \quad p_{00} = \frac{\langle \lambda^{(2)} \rangle}{\langle \lambda^{(2)} \rangle}
\]

We note that \( p_{00} \in [0, 1] \). Now let \( Y_{l} = C_{00} \text{Bin}(\left( \begin{array}{c} l \\ \lambda \end{array} \right), p_{00}) \). Then \( \mathbb{E}[Y_{l}] = \left( \begin{array}{c} l \\ \lambda \end{array} \right) \langle \lambda, \tau_{00} \rangle \) and \( \text{Var}[Y_{l}] = \left( \begin{array}{c} l \\ \lambda \end{array} \right) \langle \lambda, \eta_{00} \lambda \rangle \) and again by the central limit theorem, we have

\[
\frac{\psi_{j}(v; t) - \left( \begin{array}{c} l \\ \lambda \end{array} \right) \langle \lambda, \tau_{00} \rangle}{\sqrt{\left( \begin{array}{c} l \\ \lambda \end{array} \right) \langle \lambda, \eta_{00} \lambda \rangle}} \to \mathcal{N}(0, 1).
\]

Eq. (15) states that the locality statistics for our attributed random graphs model with \( t < t^{*} \) can be approximated by the locality statistics for an Erdős-Rényi graph with edge probability \( p_{00} \). The lemma then follows from Theorem 1.1 in [7].

**Lemma 8:** For ease of exposition we drop the index \( t^{*} \) from our discussion. Let \( \phi_{j}(v) = \psi_{j}(v) - d_{j}(v) \). Let \( M(v) \) be the number of neighbors of \( v \) that lies in \( [m] \) and \( W(v) \) be the number of neighbors of \( v \) that lies in \( [n] \setminus [m] \). The following statements are conditional on \( M(v) = l_{c} \) and \( W(v) = l_{f} \). We have

\[
\phi_{j}(v) = \sum_{k=1}^{K} \lambda_{k} \left( \begin{array}{c} l_{c} \\ k \end{array} \right) + \left( \begin{array}{c} l_{f} \\ k \end{array} \right) + \left( \begin{array}{c} l_{c} \\ k \end{array} \right)
\]

(16)
Eq. (21) states that the locality statistics \( \phi_3(v) \) for our attributed random graphs model at time \( t = t^* \) can be approximated by the locality statistics \( Y(v) \) for an unattributed kidney and egg model. The limiting distribution for the scan statistics in unattributed kidney-egg graphs had previously been considered in [7]. We provided a sketch of the arguments from [7] below, along with some minor changes to handle the case where the probability of kidney-kidney and kidney-egg connections are different.

Let \( G \) be an instance of \( \kappa(n, m, p_{11}, p_{10}, p_{00}) \), an unattributed kidney-egg graph with the probability of egg-egg, egg-kidney, and kidney-kidney connections being \( p_{11}, p_{10} \), and \( p_{00} \), respectively. \( D(v) = M(v) + W(v) \) is then the degree of \( v \) in \( G \). We now show two inequalities relating the tail distribution of \( \Delta(G) \) and \( Y(G) \). Let us define \( h(v) = \mathbb{E}[Y(v)] \), i.e.,

\[
h(v) = C_{00}p_{00}^{(D(v)^2)} + (C_{11}p_{11} - C_{00}p_{00})M(v) + (C_{10}p_{10} - C_{00}p_{00})M(v)W(v).
\]

We then have

\[
\mathbb{P}(Y(G) \geq a_{n,m}) = \mathbb{P}\left( \bigcup_{v \in V(G)} Y(v) \geq a_{n,m} \right) \\
= \mathbb{P}\left( \bigcup_{v \in V(G)} Y(v) \geq a_{n,m}, D(v) \geq a_{n,m} \right) \\
\leq P_1 + P_2
\]

where

\[
\theta_n = C_{00}\left[ \left( \frac{C}{S} \right) p_{00}(1 - p_{00}) \right]^{1/2} \log n \\
P_1 = \mathbb{P}\left( \bigcup_{v \in V(G)} D(v) \geq a_{n,m}, h(v) \geq a_{n,m} - \theta_n \right) \\
P_2 = \mathbb{P}\left( \bigcup_{v \in V(G)} D(v) \geq a_{n,m}, Y(v) - h(v) \geq \theta_n \right).
\]

We now show that \( P_2 \) is negligible as \( n \to \infty \). To proceed, let \( A \) be the event \( |M(v)| = e, W(v) = f \) and let \( p_{e,f} = \mathbb{P}(A) \). \( P_2 \) can then be bounded as follows

\[
\frac{P_2}{n} \leq \sum_{e + f \geq a_{n,m}} \mathbb{P}(Y(v) - h(v) \geq \theta_n | A)p_{e,f} \\
= \sum_{e + f \geq a_{n,m}} \mathbb{P}\left( Y(v) - h(v) \geq \frac{\theta_n}{\text{Var}[Y(v)]^{1/2}} | A \right) p_{e,f} \\
\leq \sum_{e + f \geq a_{n,m}} (1 + o(1))\mathbb{P}(Z \geq \Theta(\log n))p_{e,f} \\
= o(n^{-1}).
\]

We now consider \( P_1 \). We note that \( P_1 \leq R_1 + R_2 \) where

\[
R_1 = \mathbb{P}\left( \bigcup_{v \in [m]} D(v) \geq a_{n,m}, h(v) \geq a_{n,m} - \theta_n \right) \\
R_2 = \mathbb{P}\left( \bigcup_{v \in [m]} D(v) \geq a_{n,m}, h(v) \geq a_{n,m} - \theta_n \right).
\]

Let us define \( g(v) = h(v) - C_{00}p_{00}^{(D(v)^2)} \). \( R_1 \) is then bounded as follows

\[
R_1 \leq \mathbb{P}\left( \bigcup_{v \in [m]} h(v) \geq a_{n,m} - \theta_n \right) \\
\leq \mathbb{P}\left( \bigcup_{v \in [m]} D(v) \geq \sqrt{\frac{3(a_{n,m} - \theta_n - g(v))}{C_{00}p_{00}}} \right).
\]

We now consider the term \( a_{n,m} - g(v) \). We have

\[
a_{n,m} - g(v) = C_{00}p_{00}^{(N(v)^2)} + (C_{11}p_{11} - C_{00}p_{00})(\frac{C}{S}) - \frac{M(v)^2}{2} + (C_{10}p_{10} - C_{00}p_{00})|\mu_{\xi}| - M(v)W(v).
\]

Let \( \mathcal{E} \) and \( \mathfrak{g} \) be sets of vertices defined by

\[
\mathcal{E} = \{ v : |M(v) - \mu_{\xi}| \leq \sigma_{\xi} \log m \} \\
\mathfrak{g} = \{ v : |W(v) - \mu_{\xi}| \leq \sigma_{\xi} \log(n - m) \}.
\]

Then we have, for \( v \in \mathcal{E} \cap \mathfrak{g} \)

\[
a_{n,m} - g(v) = C_{00}p_{00}^{(N(v)^2/2)} + \Theta(m^{3/2} \log m) \\
+ \Theta(m \sqrt{n - m}).
\]
When $m = \Omega(\sqrt{n \log n})$, Eq. (27) gives
\[
a_{n,m} - g(v) = N_k^2 \left( \frac{C_{70} \log m}{2} + O(n^{-1/2 - a} \log n) \right).
\]  
(28)
for some $a > 0$. The set $\{v \in [m]\}$ can be partition into $\{v \in [m] \setminus (\mathcal{E} \cap \mathcal{G})\}$ and $\{v \in [m] \setminus (\mathcal{E} \cap \mathcal{F})\}$. We can show that $\mathbb{P}\{v \in [m] \setminus (\mathcal{E} \cap \mathcal{G})\} = o(1)$ by using a concentration inequality, e.g., Hoeffding’s bound. We thus have
\[
R_1 \leq \mathbb{P}\left( \bigcup_{v \in [m]} D(v) \geq N_k \sqrt{1 + O(\frac{\log n}{\log \log n})} \right) + o(n^{-1})
= \mathbb{P}\left( \bigcup_{v \in [m]} D(v) \geq N_k + O(n^{1/2 - a} \log n) \right) + o(n^{-1})
= \mathbb{P}\left( \Delta \geq \mu_{E + F} + \sigma_{E + F} (z_m + O(\frac{\log n}{\log n})) \right) + o(n^{-1})
\rightarrow \mathbb{P}(\Delta \geq N_k).
\]  
(29)
The same argument can be applied to $R_2$ to show that
\[
R_2 \leq \mathbb{P}\left( \bigcup_{v \in [n] \setminus [m]} D(v) \geq N_k (1 + o(1)) \right) = o(1).
\]  
(30)
Eq. (22) is therefore established.

Eq. (23): We start by noting that
\[
\mathbb{P}(Y(G) \geq a_{n,m}) = \mathbb{P}\left( \bigcup_{v \in [n]} Y(v) \geq a_{n,m} \right)
\geq \mathbb{P}\left( \bigcup_{v \in [m]} Y(v) \geq a_{n,m}, D(v) \geq N_k \right)
\geq \mathbb{P}\left( \bigcup_{v \in [m]} D(v) \geq N_k \right)
- \mathbb{P}\left( \bigcup_{v \in [n]} Y(v) < a_{n,m}, D(v) \geq N_k \right).
\]  
We now show that $\mathbb{P}(\bigcup_{v \in [m]} Y(v) < a_{n,m}, D(v) \geq N_k) \to 0$ as $n \to \infty$. Let $v \in [m]$ be arbitrary. It is then sufficient to show that $m \mathbb{P}(Y(v) < a_{n,m}, D(v) \geq N_k) = o(1)$. We note that $\mathbb{P}(Y(v) < a_{n,m}, D(v) \geq N_k)$ can be rewritten as
\[
\sum_{e+f \geq N_k} \mathbb{P}(Y(v) \leq a_{n,m}, M(v) = e, W(v) = f) p_{e,f}.
\]  
(31)
We now split the indices set $e+f \geq N_k$ in Eq. (31) into three parts $S_1$, $S_2$ and $S_3$, namely
\[
S_1 = \{ e \geq \mu_E + \sigma_E \log m \}
\]  
(32)
\[
S_2 = \{ e \leq \mu_E + \sigma_E \log m, e + f \geq N_k + \varphi(n) \}
\]  
(33)
\[
S_3 = \{ e < \mu_E + \sigma_E \log m, e + f \geq N_k + \varphi(n) \}
\]  
(34)
where $\varphi(n) = \Theta(n^{1/2 - a})$ for some $a > 0$. We can then show that $m \mathbb{P}(M(v) = e, W(v) = f, \{e,f\} \in S_1) = o(1)$ by applying a concentration inequality. Similarly, $e + f \geq N_k$ and $e \leq \mu_E + \sigma_E \log m$ implies that
\[
f \geq \mu_F + (z_m - o(1)) \sigma_F
\]  
(35)
and once again, by a concentration inequality, we can show that $m \mathbb{P}(M(v) = e, W(v) = f, \{e,f\} \in S_2) = o(1)$. As for $S_3$, from the fact that $e + f \geq N_k + \varphi(n)$, we have the bound
\[
a_{n,m} - h(v) \leq (C_{11} p_{11} - C_{10} p_{10}) \lfloor m \sigma_E \log m + \frac{(\log m)}{2} \rfloor
- C_{09} p_{09} N_k \varphi(n).
\]  
(36)
As $\text{Var}[Y(v)] = \Theta(N_k)$ for $\{M(v), W(v)\} \in S_3$, we have
\[
p_{S_3} = \sum_{\{e,f\} \in S_3} \mathbb{P}(Y(v) < a_{n,m}) p_{e,f}
\leq \sum_{\{e,f\} \in S_3} \mathbb{P}(Y(v) < a_{n,m}) \mathbb{P}(Y(v) \leq \frac{\mu_E - h(v)}{\text{Var}(Y(v))^{1/2}}) p_{e,f}
\leq \sum_{\{e,f\} \in S_3} \mathbb{P}(Z \leq \frac{m^{1/2} \log m - \varphi(n)}{N_k}) p_{e,f}.
\]  
(37)
We now set $a = \frac{1}{2(k+1)}$. Then for $m = O(n^{k/(k+1)})$ and $\varphi(n) = O(n^{1/2 - a})$ we have
\[
m^{1/2} \log m - \varphi(n) = -O(n^{k/(2(k+1))})
\]  
(38)
which then implies
\[
m p_{S_3} \leq m \sum_{\{e,f\} \in S_3} \mathbb{P}(Z \leq -O(n^{k/(2(k+1))}) p_{e,f}.
\]  
(39)
Thus $\mathbb{P}(Y(v) < a_{n,m}, D(v) \geq N_k) \to 0$ as desired.

From Eq. (22) and Eq. (23), we have
\[
\lim \mathbb{P}(Y(G) \geq a_{n,m}) = \lim \mathbb{P}(\Delta(G) \geq N_k).
\]  
(40)
Let $N_{k,y} = N_k + y \frac{\sigma_{E + F}}{\sqrt{2 \log m}}$. We now define $a_{n,m,y}$ as
\[
(a_{n,m,y} - \mu_{E + F}) / \sigma_{E + F} \left( N_k + y \frac{\sigma_{E + F}}{\sqrt{2 \log m}} \right).
\]
The above expression is equal to
\[
a_{n,m} + (\lambda, \pi_{00}) \sigma_{E + F} / \sqrt{2 \log m} \left( N_k + y \frac{\sigma_{E + F}}{\sqrt{2 \log m}} \right) + O(1).
\]  
(41)
We thus have
\[
a_{n,m,y} = a_{n,m} + (y + o(1)) b_{n,m}.
\]
We therefore have
\[
\lim \mathbb{P}(Y(G) \geq a_{n,m}) = \lim \mathbb{P}(\frac{\Delta(G) - a_{n,m}}{b_{n,m}} \geq y)
= \lim \mathbb{P}(\Delta(G) \geq N_{k,y})
= \lim \mathbb{P}(\frac{\Delta(G) - N_k}{\sigma_{E + F}} \geq \frac{y}{\sqrt{2 \log m}}).
\]
Because $\Delta(G)$ converges weakly to a Gumbel distribution in the limit ([3], [7]), we have
\[
\mathbb{P}(\frac{\Delta(G) - a_{n,m}}{b_{n,m}} \leq y) \to e^{-e^{-y}}.
\]  
(42)

References

