Generalizing the Mann-Whitney-Wilcoxon Statistic

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Abstract

We present a generalization of the classical Mann-Whitney-Wilcoxon (MWW) statistic for testing stochastic ordering between two distributions. Our class of statistics includes as special cases the generalizations of the MWW statistic by Kochar (1978), Deshpande and Kochar (1980), Stephenson and Ghosh (1985), Ahmad (1996) and Priebe and Cowen (1998). We establish the asymptotic normality and prove the admissibility of the generalization in the sense of Pitman's asymptotic efficacy. Corresponding distribution-free confidence intervals and Hodges-Lehmann estimators are derived and a generalization of the Wilcoxon signed rank statistic for testing the center of a symmetric, univariate, continuous distribution is obtained.

KEY WORDS: Mann-Whitney; Wilcoxon; U-statistic; Distribution-free confidence interval; Hodges-Lehmann estimator; Pitman's asymptotic efficacy.

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1 Introduction

The pioneering papers of Wilcoxon (1945) and Mann and Whitney (1947) mark the beginning of modern nonparametric statistics. Wilcoxon (1945) provided a test statistic, the Wilcoxon rank sum statistic, which stimulated the development of rank-based nonparametric methods. Mann and Whitney (1947) introduced an equivalent statistic in the form of what has become known as a U-statistic, a class of unbiased estimators of characteristics (parameters) of a population (or populations). The study of U-statistics has attracted a great deal of attention ever since (see, for instance, Serfling 1980, Behnen 1989). This article presents a generalization of the classical Mann-Whitney-Wilcoxon (MWW) statistic arising from the Mann and Whitney U-statistic representation.

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ be (jointly) independent random samples from distribution functions $F(x)$ and $G(y)$, respectively. Consider fixed $r < n$ and $s < m$ and subsamples $X_1, \cdots, X_r$ and $Y_1, \cdots, Y_s$ (without loss of generality), and denote by $X_{1:r} \leq X_{2:r} \cdots \leq X_{r:r}$ and $Y_{1:s} \leq Y_{2:s} \cdots \leq Y_{s:s}$ the order statistics obtained from these subsamples. The problem of interest is to test $H_0 : F(x) = G(x)$ for every x versus $H_1 : F(x) \geq G(x)$ for every x, with strict inequality for at least one x; that is, X is stochastically smaller than Y.

The MWW statistic, $\frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} I_{\{X_i < Y_j\}}$ in Mann and Whitney's formulation, is an unbiased estimator of the probability parameter $P(X_{1:1} < Y_{1:1})$ and is one of the most frequently used distribution-free statistics for testing stochastic ordering. Kochar (1978), Deshpande and Kochar (1980), Stephenson and Ghosh (1985) and Ahmad (1996) generalized the classical MWW statistic to an unbiased estimator of the probability parameter $P(X_{r:r} < Y_{s:s})$ and this generalization achieves higher efficacy for some continuous distributions. The generalization of Priebe and Cowen (1998), an estimator of $P(X_{1:r} < Y_{1:s})$, also achieves higher efficacy than MWW for some continuous distributions. These findings motivate our further generalization of the classical MWW statistic to estimators of the class of probability parameters $P(X_{k:r} < Y_{l:s})$ for each $1 \leq k \leq r$ and $1 \leq l \leq s$.

This paper investigates some statistical properties of our generalized MWW ($GMWW$) statistic such as asymptotic normality, admissibility and confidence intervals. A similar generalization is provided, in the one-sample case, for the Wilcoxon signed rank statistic.

2 Rational

The development that follows is fundamental to our generalization of the MWW statistic. Let X and Y be two independent random variables and consider testing $H_0 : F(x) = G(x)$ for all x versus $H_1 : F(x) \geq G(x)$ for all x

with strict inequality for at least one x; that is, $X \overset{st}{\leq} Y$. The MWW statistic is based on the functional

$$\gamma = P(X_{1:1} < Y_{1:1}) = \int_{-\infty}^{\infty} F(x)dG(x)$$

which is equal to $\frac{1}{2}$ under H_0 and is strictly larger than $\frac{1}{2}$ under H_1. Thus large values of γ imply that H_1 is true.

Note that $X \overset{st}{\leq} Y$ is equivalent to $u(F(x)) \geq u(G(x))$ where the real-valued function $u(x)$ defined on $[0, 1]$ is bounded and strictly increasing in x. The MWW probability parameter γ is but one in a general class of probability parameters that can be used to perform our test of stochastic ordering. Namely, let $v(x)$ be another real-valued, increasing, and continuous function defined on $[0, 1]$ and define

$$\gamma^{(u,v)} = \int_{-\infty}^{\infty} u(F(x))dv(G(x)). \tag{1}$$

Under H_0, $\gamma^{(u,v)} = \int_0^1 u(x)dv(x) \equiv \gamma_0^{(u,v)}$, while under H_1, $\gamma^{(u,v)} > \gamma_0^{(u,v)}$. (The functions u and v must be chosen so that $|\gamma^{(u,v)}| < \infty$).

We now consider generalizing the conventional MWW functional γ to the class where u and v can be defined in terms of distribution functions of the order statistics. Suppose that $X_1 \cdots X_r (Y_1 \cdots Y_s)$ are $r(s)$ independent copies of $X(Y)$ and $X_{1:r} \cdots X_{r:r}(Y_{1:s} \cdots Y_{s:s})$ are the order statistics obtained by arranging the preceding random sample in increasing order of magnitude. Define the $GMWW$ probability parameter of interest

$$\gamma^{(k:r,l:s)} = P(X_{k:r} < Y_{l:s}) = \int_{-\infty}^{\infty} F_{k:r}(x) dG_{l:s}(x)$$

where $F_{k:r}(x) = \sum_{i=k}^{r} \binom{r}{i} F^i(x)(1 - F(x))^{r-i}$, $G_{l:s}(x) = \sum_{i=l}^{s} \binom{s}{i} G^i(x)(1 - G(x))^{s-i}$, and where k, r, l and s are four nonnegative integers. For ease of notation, it is assumed that $F_{k:r}(x) = 0$ whenever $k > r$ and $G_{l:s}(x) = 0$ whenever $l > s$, and that $F_{0:r}(x) = 1$ whenever $r > 0$ and $G_{0:s}(x) = 1$ whenever $s > 0$. As before, notice that $\gamma^{(k:r,l:s)} \geq \frac{\sum_{i=k}^{r} \binom{l+i-1}{i}\binom{r+s-i-l}{r-i}}{\binom{r+s}{r}}$ with equality if and if H_0 holds.

The empirical estimate of $\gamma^{(k:r,l:s)}$ is given by

$$\begin{aligned} GMWW_{n,m}^{(k:r,l:s)} &= \frac{\sum_C I(X_{k:r}(X_{i_1}, \cdots, X_{i_r}) < Y_{l:s}(Y_{j_1}, \cdots, Y_{j_s}))}{\binom{n}{r}\binom{m}{s}} \\ &= \frac{\sum_{i=k}^{n-r+k} \sum_{j=l}^{m-s+l} \binom{i-1}{k-1}\binom{n-i}{r-k}\binom{j-1}{l-1}\binom{m-j}{s-l} I_{\{X_{(i)}<Y_{(j)}\}}}{\binom{n}{r}\binom{m}{s}} \end{aligned} \quad (2)$$

where k, l, r, and s are fixed and $k \leq r \ll n$, $l \leq s \ll m$. The summation notation $\sum_C$ extends over all indices $1 \leq i_1 < \cdots < i_r \leq n$, $1 \leq j_1 < \cdots < j_s \leq m$. $X_{(1)}, \cdots, X_{(n)}(Y_{(1)}, \cdots, Y_{(m)})$ denote the order statistics of the random sample $X_1, \cdots, X_n(Y_1, \cdots, Y_m)$ to distinguish these order statistics from the subsample order statistics $X_{k:r}(Y_{l:s})$.

Remarks:

1. The classical MWW statistic is simply $GMWW_{n,m}^{(1:1,1:1)}$, the Kochar-Ahmad statistic is $GMWW_{n,m}^{(r:r,s:s)}$, and the Priebe & Cowen statistic

is $GMWW_{n,m}^{(1:r,1:s)}$.

2. The representation (2) makes the calculation of $GMWW_{n,m}^{(r:r,s:s)}$ feasible since it dramatically reduces the computational task compared to the U-statistic representation.

3. $F_{k:r}(x)$ and $G_{l:s}(x)$ are increasing bounded functions in $F(x)$ and $G(x)$, respectively. Therefore, the functional $\gamma^{(k:r,l:s)}$ associated with $GMWW_{n,m}^{(k:r,l:s)}$ is a special case of functional $\gamma^{(u,v)}$. The further generalization to functional (1) would be of interest as well.

4. The admissibility of $GMWW_{n,m}^{(r:r,s:s)}$ for the test of stochastic ordering is investigated in the sense of Pitman's asymptotic efficacy.

5. The asymptotic behavior of the statistic $GMWW_{n,m}^{(k:r,l:s)}$ is summarized in the following theorem.

Theorem 1 *Let $n \to \infty$ and $m \to \infty$ such that $(\frac{n}{n+m}) \to \lambda \in (0,1)$. Fix $1 \leq k \leq r$ and $1 \leq l \leq s$. Then $(n+m)^{1/2}(GMWW_{n,m}^{(k:r,l:s)} - \gamma^{(k:r,l:s)})$ is asymptotically normal with mean 0 and variance given by*

$$\begin{aligned}\sigma^2_{k:r,l:s} &= \frac{r^2}{\lambda} var(\int_{-\infty}^{X_1}(F_{k:r-1} - F_{k-1:r-1})(x)dG_{l:s}(x)) + \\ &\quad \frac{s^2}{(1-\lambda)} var(\int_{-\infty}^{Y_1}(G_{l:s-1} - G_{l-1:s-1})(x)dF_{k:r}(x)).\end{aligned}$$

Under H_0, $\sigma^2_{k:r,l:s}$ reduces to

$$\frac{1}{\lambda(1-\lambda)}\{[\frac{r\binom{k+l-2}{k-1}\binom{r+s-k-l}{r-k}}{\binom{r+s-1}{s}}]^2 - [\frac{\binom{k+l-2}{l-1}\binom{r+s-k-l+1}{r-k}(s-l+1)}{\binom{r+s}{r}}]^2 -$$
$$\frac{2r^2\binom{l+k-2}{l-1}\binom{r+s-k-l}{s-l}\binom{r-1}{k-1}s!}{\binom{r+s-1}{s}(l-1)!(s-l)!}\sum_{i=k+l-1}^{r+s-1}$$
$$\binom{r+s-1}{i}\frac{(l+i+k-1)!(2r+2s-i-l-k-1)!}{(2r+2s-1)!}\}.$$

The proof of Theorem 1 is included in the Appendix.

3 Pitman's asymptotic efficacy analysis.

Pitman's asymptotic efficacy (PAE) is often used for the purpose of comparing two test procedures on the basis of their asymptotic distributional properties (see Pitman 1979). It is defined in our case as follows:

$$PAE(\gamma^{(k:r,l:s)}) = \{\frac{d}{d\theta}\gamma_\theta^{(k:r,l:s)}\}^2|_{\theta\to\theta_0}/\sigma^2_{k:r,l:s}$$

where $F = F_{\theta_0}$ and $G = F_\theta$, with $\theta = \theta_0 + c/n^{1/2}$. Applying this definition we easily see that for the location problem $G(x) = F(x-\theta)$, under the assumption the density of F exists and is equal to f,

$$\frac{d}{d\theta}\gamma_\theta^{(k:r,l:s)}|_{\theta\to 0} = \frac{r!s!}{(k-1)!(r-k)!(l-1)!(s-l)!}\int_{-\infty}^{\infty} F^{k+l-2}(1-F)^{r+s-k-l}f^2(x)dx.$$

Next, let us investigate some commonly used distributions in the location case to obtain the choice of k, r, l, and s giving maximum efficacy. We shall discuss the double exponential, uniform, exponential, and logistic distributions. For clarity, PAEs displayed below are scaled by the constant $\lambda(1-\lambda)$.

1. The Double Exponential: $f(x) = 1/2exp(-|x|)$, where $-\infty < x < \infty$.

Table 1: $PAEs$ under the double exponential.

(r,s)	*(r/2:r,s/2:s)*	*(r:r,s:s)*	*(1:r,1:s)*	best
(1,1)	.75	.75	.75	.75
(2,2)	.804	.595	.595	.804
(3,3)	.833	.413	.413	.833
(4,4)	.759	.301	.301	.852
(5,5)	.866	.234	.234	.866
(6,6)	.821	.190	.190	.876
(7,7)	.885	.159	.160	.885
(8,8)	.854	.138	.138	.891
(9,9)	.897	.121	.121	.897
(10,10)	.874	.108	.108	.902
(19,19)	.927	.054	.054	.927
(29, 29)	.941	0.	0.	.941
(35,35)	.945	0.	0.	.945

It is evident from Table 1 that with the same subsample sizes r and s, the statistic that maximizes PAE among $GMWW$ statistics is superior to the conventional MWW, the Kochar-Ahamd, and the Priebe & Cowen statistics; some members of the $GMWW$ class other than these three gain the highest efficacy. This fact proves the admissibility of the $GMWW$. In particular, notice that the median selection, $MMWW^{(r,s)} = GMWW^{([r/2]:r,[s/2]:s)}$ where $[x]$ is the largest integer less than or equal to x, is much better than the Kochar-Ahmad and Priebe & Cowen statistics although it is not necessarily the best. (While it appears that $PAEs$ for Kochar-Ahmad and Priebe & Cowen statistics are identical in this example, this is not, in general, the case.) As subsample sizes r, s get large, the best GMWW statistic tends to be optimal. In other word, it approaches the best efficacy of 1.

2. The Uniform $[0, 1]$: $f(x) = 1$, $0 \leq x \leq 1$.

For this case, $PAE(k : r, l : s)$ is a function of $r+s$ and $k+l$. For the sake of illustration, consider subsample sizes $r = s = 10$. We obtain PAE(1:1,1:1)=12 and $PAE(1 : 10, 1 : 10) = PAE(10 : 10, 10 : 10) = 43.2 \geq PAE(i : 10, j : 10)$ for $1 \leq i \leq 10$ and $1 \leq j \leq 10$. Thus, the MWW is not the best statistic; the Priebe & Cowen statistic and the Kochar-Ahmad statistic are admissible relative to the MWW statistics for $r = s = 10$.

Table 2 : $PAEs$ under the uniform distribution.

(r,s)	*(r/2:r,s/2:s)*	*(r:r,s:s)*	*(1:r,1:s)*	*best*
(1,1)	12	12	12	12
(2,2)	8.2	12.4	12.4	12.4
(3,3)	7.05	15.8	15.8	15.8
(4,4)	7.01	19.59	19.59	19.59
(5,5)	6.09	23.45	23.45	23.45
(6,6)	6.05	27.37	27.37	27.37
(7,7)	5.66	31.3	31.3	31.3
(8,8)	5.63	35.3	35.3	35.3
(9,9)	5.41	39.2	39.2	39.2
(10,10)	5.38	43.2	43.2	43.2
(20,20)	4.87	83.1	83.1	83.1
(30,30)	4.68	123.1	123.1	123.1

3. The Exponential: $f(x) = exp(-x), x \geq 0$.

For illustrative purposes, consider subsample size $r = s = 5$. Since $PAE(1:1,1:1) = 3$ and $PAE(1:5,1:5) = 19 > PAE(5:5,5:5) = 0.235$, the Priebe & Cowen statistic is admissible relative to the Kochar-Ahmad and the MWW statistics. Furthermore, $PAE(2:5,2:5) = 3.71$; this $GMWW^{(2:5,2:5)}$ statistic is superior to the Kochar-Ahmad and MWW statistics, whereas it is inferior to the Priebe & Cowen statistic.

Table 3 : $PAEs$ under the exponential distribution.

(r,s)	*(r/2:r,s/2:s)*	*(r:r,s:s)*	*(1:r,1:s)*	*best*
(1,1)	3	3	3	3
(2,2)	2.0	0.78	7	7
(3,3)	1.7	0.44	11	11
(4,4)	2.7	0.30	15	15
(5,5)	1.5	0.23	19	19
(6,6)	2.05	0.19	23	23
(7,7)	1.4	0.16	27	27
(8,8)	1.78	0.14	31	31
(9,9)	1.35	0.12	35	35
(10,10)	1.63	0.11	39	39
(20,20)	1.34	0.05	79	79
(30,30)	1.25	0.03	119	119

4. The Logistic: $f(x) = exp(-x)/(1 + exp(-x))^2$, where $-\infty < x < \infty$.

Note that $PAE(1:1, 1:1) = 1/3$ and $PAE(1:10, 1:10) = PAE(10:10, 10:10) = 0.088$ and that $PAE(5:10, 5:10) = 0.299$. Although the $GMWW^{(5:10,5:10)}$ statistic is inferior to the conventional MWW, it outperforms the Kochar-Ahmad and Priebe & Cowen statistics when $s = r = 10$.

Table 4: PAEs under the logistic distribution

(r,s)	*(r/2:r,s/2:s)*	*(r:r,s:s)*	*(1:r,1:s)*	*best*
(1,1)	1/3	1/3	1/3	1/3
(2,2)	.329	.28	.28	.329
(3,3)	.323	.224	.224	.323
(4,4)	.304	.185	.185	.319
(5,5)	.315	.157	.157	.315
(6,6)	.304	.136	.136	.311
(7,7)	.308	.12	.12	.308
(8,8)	.302	.107	.107	.305
(9,9)	.303	.097	.097	.303
(10,10)	.299	.088	.088	.301

4 Extensions

Consider the independent random samples $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ from continuous distributions with c.d.f.'s $F(x)$ and $F(x-\theta)$, respectively. A class of Hodges-Lehmann estimators for θ, using $GMWW_{n,m}^{(k:2k-1,l:2l-1)}$ that satisfies the three conditions of Randles and Wolfe (1979, p.204), is attained as follows:

$$\begin{aligned}\hat{\theta}_{n,m}^{(k:2k-1,l:2l-1)} &= median\{X_{k:2k-1}(X_{i_1}, \cdots, X_{i_{2k-1}}) - Y_{l:2l-1}(Y_{j_1}, \cdots, Y_{j_{2l-1}}), \\ &\quad 1 \leq i_1 \leq \cdots \leq i_{2k-1} \leq n, 1 \leq j_1 < \cdots < j_{2l-1} \leq m\}\end{aligned}$$

$$= D_{([L/2])}$$

where $1 \le i_1 \le \cdots \le i_{2k-1} \le n$, and $1 \le j_1 < \cdots < j_{2l-1} \le m$, and where $D_{(1)} \le \cdots \le D_{(L)}$ are the ordered values of differences $X_{k:2k-1}(X_{i_1}, \cdots, X_{i_{2k-1}})$ $-Y_{l:2l-1}(Y_{j_1}, \cdots, Y_{j_{2l-1}})$, $1 \le i_1 < \cdots < i_{2k-1} \le n$, $1 \le j_1 < \cdots < j_{2l-1} \le m$, and $L = \binom{n}{2k-1}\binom{m}{2l-1}$. For $s = 1, \cdots, L$, $D_{(s)}$ takes values from the set $\{D_{ij} = X_{(i)} - Y_{(j)}, i = k, \cdots, n-r+k, \quad and \quad j = l, \cdots, m-s+l\}$. If $D_{(s)} = D_{i_s j_s}$, then the number of duplicates of $D_{i_s j_s}$ in the sequence $\{D_{(s)}, s = 1, \cdots, L\}$ is $\binom{i_s-1}{k-1}\binom{n-i_s}{k-1}\binom{j_s-1}{l-1}\binom{m-j_s}{l-l}$. These observations make the calculation of $\hat{\theta}_{n,m}^{(k:2k-1,l:2l-1)}$ much easier.

An approximate distribution-free confidence interval for a parameter θ can be derived from a distribution-free statistic, say V, for testing $H_0 : \theta = 0$. For $i = 1, \cdots, \binom{n}{r}$ and $j = 1, \cdots, \binom{m}{s}$, define $D_{ij} = Y_{l:s}(Y_{j_1}, \cdots, Y_{j_s}) - X_{k:r}(X_{i_1}, \cdots, X_{i_r})$. where $1 \le j_1 < \cdots < j_s \le m$ and $1 \le i_1 < \cdots < i_r \le n$. Thus, $V_{n,m}^{(k:r,l:s)} = (\# \quad of \quad positive \quad D_{ij})$. Let $D_{(1)} \le \cdots \le D_{(R)}$ be the ordered D_{ij} differences, where $R = \binom{n}{r}\binom{m}{s}$. Then $[D_{(v+1)}, D_{(R-v)})$ is an approximate $100(1-\alpha)$ percent asymptotic distribution-free confidence interval for θ, where $v(\alpha/2, k, l, r, s, n, m)$ is the integer closest to $(\frac{\sum_{i=k}^{r}\binom{l+i-1}{i}\binom{r+s-i-l}{r-i}}{\binom{r+s}{r}} - z_{\alpha/2}\sigma_{k:r,l:s}/(m+n)^{1/2}) \times R$ and $\sigma^2_{k:r,l:s}$ is the null variance for $GMWW_{n,m}^{(k:r,l:s)}$ presented in Theorem 1. Again, the calculation of $V_{n,m}^{(k:r,l:s)}$ can be simplified as above.

5 One-sample generalization.

Let $X_1, X_2, \cdots, X_n$ be independent random variables with continuous c.d.f. F. Assume that the distribution F is symmetric about a point θ such that for each x, $F(x+\theta) + F(-x+\theta) = 1$; θ is referred to as the center of the distribution. This is equivalent to saying that $u(F(x+\theta)) = u(\bar{F}(-x+\theta))$, where u is an arbitrary strictly increasing, bounded, real-valued function on $[0,1]$ and $\bar{F}(x) = 1 - F(x)$. Consider the problem of testing that the center of symmetry, θ, is 0 against $\theta \neq 0$. The Wilcoxon signed rank (WSR) statistic used for this purpose is, essentially, a nonparametric estimator of the functional $\delta = \int_{-\infty}^{\infty} \bar{F}(-x) dF(x)$. It is clear that δ is but a special case of the class of functionals $\delta^{(u,v)}$ defined by $\delta^{(u,v)} = \int_{-\infty}^{\infty} u(\bar{F}(-x)) dv(F(x))$ where the real-valued function v on $[0,1]$ is increasing, continuous and bounded. Without loss of generality, under $H_1 : \theta > 0$,

$$\delta^{(u,v)} = \int_{-\infty}^{\infty} u(F(x+2\theta)) dv(F(x))$$

is larger than under H_0, for which $delta^{(u,v)} = \int_{-\infty}^{\infty} u(F(x)) dv(F(x))$. Thus, the associated test rejects for large values of $\delta^{(u,v)}$. Proceeding as before, one special case of functional $\delta^{(u,v)}$ is

$$\delta^{(k:r,l:s)} = P(X_{k:r}(X_1, \cdots, X_r) + X_{l:s}(X_{r+1}, \cdots, X_{r+s}) > 0)$$

where k, r, l, and s are fixed.

In fact, $\delta^{(k:r,l:s)}$ can be estimated empirically by

$$GWSR_n^{(k:r,l:s)} = \frac{\sum_c I(X_{k:r}(X_{i_1}, \cdots, X_{i_r}) + X_{l:s}(X_{i_{r+1}}, \cdots, X_{i_{r+s}}) > 0)}{\binom{n}{r+s}\binom{r+s}{r}}$$
$$= \frac{\sum_{i=k}^{n-r+k} \sum_{j=l}^{n-s+l} w_{ij} I_{\{X_{(i)}+X_{(j)}<0\}}}{\binom{n}{r+s}\binom{r+s}{r}}$$

where $\sum_c$ stands for summation over all permutations $(i_1, \cdots, i_{r+s})$ of $r+s$ integers such that $1 \le i_1 < i_2 < \cdots < i_r \le n$, $1 \le i_{r+1} < i_{r+2} < \cdots < i_{r+s} \le n$, and $i_e \ne i_f$ if $e \ne f$, $1 \le e \le r$ and $1 \le f \le s$, and where for $i < j$,

$$w_{ij} = \sum_{v=0}^{l-1} \binom{i-1}{k-1}\binom{i-k}{v}\binom{j-i-1}{l-v-1}\binom{n-j}{s-l}\binom{n+v-i-s}{r-k}$$

$w_{ij} = 0$ for $\quad i = j$

$$w_{ij} = \sum_{v=0}^{k-1} \binom{j-1}{l-1}\binom{j-l}{v}\binom{i-j-1}{k-v-1}\binom{n-i}{r-k}\binom{n+v-j-r}{s-l}$$

for $i > j$. $GWSR_n^{(k:r,l:s)}$ includes the WSR statistic and the one-sample Ahmad statistic (1996) as special cases, $GWSR_n^{(1:1,1:1)}$ and $GWSR_n^{(1:r,1:1)}$, respectively. The following theorem summarizes the asymptotic behavior of $GWSR_n^{(k:r,l:s)}$.

Theorem 2 *Let $n \to \infty$. Fix $1 \le k \le r < \infty$ and $1 \le l \le s < \infty$. Then $n^{1/2}(GWSR_n^{(k:r,l:s)} - \delta^{(k:r,l:s)})$ is asymptotically normal with mean 0 and variance $\sigma^2_{k:r,l:s}$ given by*

$$var\{r[\int_{-X_1}^{\infty} \bar{F}_{r-k:r-1}(-x)dF_{l:s}(x) + \int_{-\infty}^{-X_1} \bar{F}_{r-k+1:r-1}(-x)dF_{l:s}(x)] +$$
$$s[\int_{-\infty}^{X_1} F^{l-1:s-1}(x)d\bar{F}_{r-k+1:r}(-x) - \int_{-\infty}^{X_1} F^{l:s-1}(x)d\bar{F}_{r-k+1:r}(-x)]\}.$$

Under $H_0 : F(x) + F(-x) = 1$ *for all* x*, the variance reduces to*

$$[\frac{r!s!(r+l-k-1)!(s+k-l-1)!}{(k-1)!(r-k)!(l-1)!(s-l)!(r+s-1)!}]^2 \times$$
$$(1 - \frac{2(s+r-1)!}{(s-l+k-1)!(r+l-k-1)!} \sum_{i=r+l-k}^{s+r-1} \frac{\binom{s+r-1}{i}}{(2r+2s-1)!}$$
$$[(2i+2l-2k-2s+1)(r+l+i-k-1)!(2s+r+k-l-i-2)! +$$
$$(2i+2k-2r-2l+1)(s-l+k+i-1)!(s+2r+l-i-k-2)!]).$$

The Pitman's asymptotic efficacy analyses of GWSR for the double exponential, uniform and logistic distribution have been performed. Similar conclusions as the two-sample case can be drawn as far as admissibility is concerned. Moreover, the corresponding Hodges-Lehmann estimators and approximate confidence intervals are obtainable as well.

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7 Appendix

Proof of Theorem 1. Following Theorem 3.4.13 of Randles and Wolfe (1979), it suffices to establish the variance. Let

$$\varphi^{(k:r,l:s)}(X_1,\cdots,X_r,Y_1,\cdots,Y_s) = I_{\{X_{k:r}(X_1,\cdots,X_r)<Y_{l:s}(Y_1,\cdots,Y_s)\}}$$

and proceed separately for four distinct cases in terms of the values of k, r, l, s.

Case 1: $2 \le k < r$ and $1 \le l \le s$.

$$\begin{aligned}
\varphi_{10}^{(k:r,l:s)}(X_1) &= E[\varphi^{(k:r,l:s)}(X_1,\cdots,X_r,Y_1,\cdots,Y_s)|X_1] \\
&= P(X_{k:r} \le Y_{l:s}|X_1) \\
&= P(X_1 \le X_{k-1:r-1}(X_2,\cdots,X_r) \le Y_{l:s}|X_1) + \\
&\quad P(X_{k-1:r-1}(X_2,\cdots,X_r) \le X_1 \le X_{k:r-1}; X_1 \le Y_{l:s}|X_1) + \\
&\quad P(X_{k:r-1}(X_2,\cdots,X_r) \le X_1 \le Y_{l:s}|X_1) + \\
&\quad P(X_{k:r-1}(X_2,\cdots,X_r) \le Y_{l:s} \le X_1|X_1) \\
&= \int_{-\infty}^{\infty} F_{k-1:r-1}dG_{l:s} + \int_{-\infty}^{X_1}(F_{k:r-1} - F_{k-1:r-1})dG_{l:s}(x).
\end{aligned}$$

Case 2: $2 \le k = r$ and $1 \le l \le s$.

$$\begin{aligned}
\varphi_{10}^{(k:r,l:s)}(X_1) &= E[\varphi^{(k:r,l:s)}(X_1,\cdots,X_r,Y_1,\cdots,Y_s)|X_1] \\
&= P(X_{r:r} \le Y_{l:s}|X_1) \\
&= P(X_1 \le X_{r-1:r-1}(X_2,\cdots,X_r) \le Y_{l:s}|X_1) +
\end{aligned}$$

$$
\begin{aligned}
& P(X_{r-1:r-1}(X_2,\cdots,X_r) \leq X_1 \leq Y_{l:s}|X_1) \\
= & \int_{X_1}^{\infty} F_{r-1:r-1}(x) dG_{l:s}(x) + \int_{-\infty}^{X_1} F_{r:r-1}(x) dG_{l:s}(x)
\end{aligned}
$$

where $F_{r:r-1}(x) = 0$.

Case 3: $1 = k < r$ and $1 \leq l \leq s$.

$$
\begin{aligned}
\varphi_{10}^{(k:r,l:s)}(X_1) &= E[\varphi^{(k:r,l:s)}(X_1,\cdots,X_r,Y_1,\cdots,Y_s)|X_1] \\
&= P(X_{k:r} \leq Y_{l:s}|X_1) \\
&= P(X_1 \leq X_{1:r-1}(X_2,\cdots,X_r) \leq Y_{l:s}|X_1) + \\
& \quad P(X_1 \leq Y_{l:s} \leq X_{1:r-1}|X_1) + \\
& \quad P(X_{1:r-1}(X_2,\cdots,X_r) \leq X_1 \leq Y_{l:s}|X_1) + \\
& \quad P(X_{1:r-1}(X_2,\cdots,X_r) \leq Y_{l:s} \leq X_1|X_1) \\
&= \int_{X_1}^{\infty} F_{0:r-1}(x) dG_{l:s}(X_1) + \int_{-\infty}^{X_1} F_{1:r-1}(x) dG_{l:s}(x)
\end{aligned}
$$

where $F_{0:r-1}(x) = 1$.

Case 4: $1 = k = r$ and $1 \leq l \leq s$.

$$
\begin{aligned}
\varphi_{10}^{(1:1,l:s)}(X_1) &= E[\varphi^{(1:1,l:s)}(X_1,Y_1,\cdots,Y_s)|X_1] \\
&= P(X_{1:1} \leq Y_{l:s}|X_1) \\
&= \int_{X_1}^{\infty} F_{0:0}(x) dG_{l:s}(y) + \int_{-\infty}^{X_1} F_{1:0}(x) dGG_{l:s}(x)
\end{aligned}
$$

where $F_{0:0}(x) = 1$ and $F_{1:0}(x) = 0$.

To unify the above results, we write

$$\varphi_{10}^{(k:r,l:s)}(X_1) \;=\; \int_{-\infty}^{\infty} F_{k-1:r-1} dG_{l:s} + \int_{-\infty}^{X_1} (F_{k:r-1} - F_{k-1:r-1}) dG_{l:s}(x)$$

where $F_{k:r}(x) = 1$ or 0 as $k = 0$ or $k > r$ and $G_{l:s}(y) = 1$ or 0 as $l = 0$ or $l > s$.

Under H_0,

$$\int_{-\infty}^{\infty} F_{k-1:r-1} dG_{l:s} \;=\; \frac{\sum_{i=k-1}^{r-1} \binom{l+i-1}{i}\binom{r+s-i-k-1}{r-i-1}}{\binom{r+s-1}{s}}$$

and

$$\int_{-\infty}^{X_1} (F_{k:r-1} - F_{k-1:r-1}) dG_{l:s}(x) \;=\; -\frac{\binom{k+l-2}{l-1}\binom{r+s-k-l}{s-l}}{\binom{r+s-1}{s}} \sum_{i=k+l-1}^{r+s-1} \binom{r+s-1}{i} F^i (1-F)^{r+s-i-1}.$$

Thus

$$\varphi_{10}^{(k:r,l:s)}(X_1) \;=\; \frac{\sum_{i=k-1}^{r-1} \binom{l+i-1}{i}\binom{r+s-i-l-1}{r-i-1}}{\binom{r+s-1}{s}} - \frac{\binom{l+k-2}{l-1}\binom{r+s-k-l}{s-l}}{\binom{r+s-1}{s}} \sum_{i=l+k-1}^{r+s-1} \binom{r+s-1}{i} F^i (1-F)^{r+s-i-1}.$$

On the other hand,

$$\begin{aligned}\varphi_{01}^{(k:r,l:s)}(X_1) &= E[\varphi^{(k:r,l:s)}(X_1, \cdots, X_r, Y_1, \cdots, Y_s)|Y_1] \\ &= 1 - [\int_{Y_1}^{\infty} G_{l-1:s-1}(x) dF_{k:r}(x) + \int_{-\infty}^{Y_1} G_{l:s-1}(x) dF_{k:r}(x)].\end{aligned}$$

Hence, under H_0,

$$\begin{aligned} E\varphi_{10}^{(k:r,l:s)}(X_1) &= E\varphi_{01}^{(k:r,l:s)}(Y_1) \\ &= \frac{\sum_{i=k}^{r} \binom{l+i-1}{i}\binom{r+s-i-l}{r-i}}{\binom{r+s}{r}}. \end{aligned}$$

Next,

$$\begin{aligned} \xi_{1,0} &= var(\varphi_{10}^{(k:r,l:s)}(X_1)) \\ &= E[\int_{-\infty}^{X_1}(F_{k:r-1} - F_{k-1:r-1})dF_{l:s}(x)]^2 - E^2\int_{-\infty}^{X_1}(F_{k:r-1} - F_{k-1:r-1})dF_{l:s}(x) \\ &= [\frac{\binom{k+l-2}{k-1}\binom{r+s-k-l}{r-k}}{\binom{r+s-1}{r}}]^2 - [\frac{\binom{k+l-2}{l-1}\binom{r+s-k-l+1}{r-k}(s-l+1)}{r\binom{r+s}{r}}]^2 - \frac{2\binom{l+k-2}{l-1}\binom{r+s-k-l}{s-l}\binom{r-1}{k-1}s!}{\binom{r+s-1}{s}(l-1)!(s-l)!} \\ &\quad \sum_{i=k+l-1}^{r+s-1}\binom{r+s-1}{i}\frac{(l+i+k-1)!(2r+2s-i-l-k-1)!}{(2r+2s-1)!}. \end{aligned}$$

Since

$$\begin{aligned} &E\int_{-\infty}^{x}(F_{k:r-1} - F_{k-1:r-1})dF_{l:s}(x) \\ &= \frac{\binom{k+l-2}{k-1}\binom{r+s-k-l+1}{r-k}(r-l+1)}{r\binom{r+s}{s}}, \end{aligned}$$

we have

$$\begin{aligned} &E[\int_{-\infty}^{X_1}(F_{k:r-1} - F_{k-1:r-1})dF_{l:s}]^2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{X_1}(F_{k:r-1} - F_{k-1:r-1})dF_{l:s}]^2 dF \\ &= [\frac{\binom{k+l-2}{l-1}\binom{r+s-k-l+1}{r-k}(s-l+1)}{r\binom{r+s}{r}}]^2 - \frac{2\binom{l+k-2}{l-1}\binom{r+s-k-l}{s-l}\binom{r-1}{k-1}s!}{\binom{r+s-1}{s}(l-1)!(s-l)!} \\ &\quad \sum_{i=k+l-1}^{r+s-1}\binom{r+s-1}{i}\frac{(l+i+k-1)!(2r+2s-i-l-k-1)!}{(2r+2s-1)!}. \end{aligned}$$

Similarly we have

$$\begin{aligned}
\xi_{0,1} &= var(\varphi_{01}^{(k:r,l:s)}(Y_1)) \\
&= [\frac{\binom{k+l-2}{l-1}\binom{r+s-k-l}{s-l}}{\binom{r+s-1}{s}}]^2 - [\frac{\binom{k+l-2}{k-1}\binom{r+s-k-l+1}{r-k}(r-k+1)}{s\binom{r+s}{s}}]^2 - \frac{2\binom{l+k-2}{k-1}\binom{r+s-k-l}{r-k}\binom{s-1}{l-1}r!}{\binom{r+s-1}{r}(k-1)!(r-k)!} \\
&\quad \sum_{i=k+l-1}^{r+s-1}\binom{r+s-1}{i}\frac{(l+i+k-1)!(2r+2s-i-l-k-1)!}{(2r+2s-1)!}.
\end{aligned}$$

Hence, $\frac{r^2\xi_{10}}{\lambda} + \frac{s^2\xi_{01}}{(1-\lambda)}$ yields the variance under H_0. Observe that the statistic $\delta_{m,n}^{(l:r,k:s)}$ is distribution free under H_0. Note also that we must choose $1 \leq k \leq r \leq m$ and $1 \leq l \leq s \leq n$.

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