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Note

On the monotone likelihood ratio property for the convolution of independent binomial random variables

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ABSTRACT

Given that r and s are natural numbers and $X \sim \text{Binomial}(r, q)$ and $Y \sim \text{Binomial}(s, p)$ are independent random variables where $q, p \in (0, 1)$, we prove that the likelihood ratio of the convolution $Z = X + Y$ is decreasing, increasing, or constant when $q < p$, $q > p$ or $q = p$, respectively.

1. Introduction

Let $r, s \in \mathbb{N}$, $z \in \{0, \ldots, r+s\}$, and $q, p \in (0, 1)$. Let $X \sim \text{Binomial}(r, q)$ and $Y \sim \text{Binomial}(s, p)$ be independent random variables. Let $P_{r,s}(z)$ denote the likelihood function of the convolution $Z = X + Y$, so that

$$P_{r,s}(z) = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{z-k} q^k (1-q)^{r-k} p^{z-k} (1-p)^{s-z+k}.$$

We will show that the ratio

$$\frac{P_{r+1,s-1}(z)}{P_{r,s}(z)}$$

is increasing, decreasing or constant – with respect to z – when $q < p$, $q > p$ or $q = p$, respectively. Moreover, this result is obtained by only appealing to elementary combinatorial identities.

This same ratio has been analyzed for its monotone likelihood ratio (MLR) properties with respect to fixed z (Ghurye and Wallace [2], Grayson [3], Huynh [4]). Our main result is that the family of convolutions of independent binomial random variables indexed by parameters r, s with $r + s = c$ constant is a MLR family in z.

In statistical inference, MLR families give rise to uniformly most powerful tests — for a given null hypothesis, the same test statistic is known to be optimal (in terms of statistical power) across an entire composite alternative hypothesis (Bickel and Doksum [1], Section 4.3). Our result demonstrates that, for $r + s = c$ constant and $q > p$, rejecting $H_0 : r = 0$ for large values of the test statistic Z is most powerful against any alternative $H_A : r = r' > 0$.

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2. Main result

Theorem. *The ratio*

$$\frac{P_{r+1,s-1}(z)}{P_{r,s}(z)}$$

is increasing, decreasing or constant – with respect to z *– when* $q < p$, $q > p$ *or* $q = p$, *respectively.*

Proof. Fix $r, s \in \mathbb{N}$ and $p, q \in (0, 1)$. We are considering the likelihood

$$P_{r,s}(z) = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{z-k} q^k (1-q)^{r-k} p^{z-k} (1-p)^{s-z+k}$$

or equivalently

$$P_{r,s}(z) = [(1-q)^r (1-p)^s] \left(\frac{p}{1-p}\right)^z S_{r,s}(z)$$

where

$$S_{r,s}(z) = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{z-k} \alpha^k$$

and $\alpha = \frac{q(1-p)}{p(1-q)}$.

In particular, for $1 \le z \le r+s$, we are interested in the difference of the likelihood ratios

$$\frac{P_{r+1,s-1}(z)}{P_{r,s}(z)} - \frac{P_{r+1,s-1}(z-1)}{P_{r,s}(z-1)} = \left(\frac{1-q}{1-p}\right)\left(\frac{S_{r+1,s-1}(z)S_{r,s}(z-1) - S_{r+1,s-1}(z-1)S_{r,s}(z)}{S_{r,s}(z)S_{r,s}(z-1)}\right).$$

Let

$$\Delta_{r,s}(z) = S_{r+1,s-1}(z)S_{r,s}(z-1) - S_{r+1,s-1}(z-1)S_{r,s}(z).$$

We will show that $\Delta_{r,s}(z)$ vanishes, is positive, or is negative when $p = q$, $q > p$, or $q < p$, respectively.

For legibility, we will use the notation $a = s-1$, $b = z-j$, $c = z-l+j$, and $d = z-k$.

First, we will rewrite the quantity $\Delta_{r,s}(z)$ in terms of powers of α, and apply an elementary combinatorial identity on selected terms:

$$\begin{aligned}\Delta_{r,s}(z) &= \sum_{j=0}^{r+1}\sum_{k=0}^{r} \binom{r+1}{j}\binom{r}{k}\left[\binom{a}{b}\binom{a+1}{d-1} - \binom{a}{b-1}\binom{a+1}{d}\right]\alpha^{j+k} \\ &= \sum_{j=0}^{r+1}\sum_{k=0}^{r} \binom{r+1}{j}\binom{r}{k}\left[\binom{a}{b}\left[\binom{a}{d-1}+\binom{a}{d-2}\right] - \binom{a}{b-1}\left[\binom{a}{d}+\binom{a}{d-1}\right]\right]\alpha^{j+k}\end{aligned}$$

and thus for each $l \in \{0, 1, \ldots, 2r+1\}$ we can express the coefficient of α^l as

$$\sum_{j=0}^{r+1} \binom{r+1}{j}\binom{r}{l-j}\left[\binom{a}{b}\left[\binom{a}{c-1}+\binom{a}{c-2}\right] - \binom{a}{b-1}\left[\binom{a}{c}+\binom{a}{c-1}\right]\right].$$

We will split this coefficient into a pair of sums

$$\sum_{j=0}^{l+1} \binom{r+1}{j}\binom{r}{l-j}\left[\binom{a}{b}\binom{a}{c-2} - \binom{a}{b-1}\binom{a}{c-1}\right] + \sum_{j=0}^{l} \binom{r+1}{j}\binom{r}{l-j}\left[\binom{a}{b}\binom{a}{c-1} - \binom{a}{b-1}\binom{a}{c}\right]$$

and separately analyze each sum in this pair of sums.

Note that twice the first of these sums can be expressed as

$$\sum_{j=0}^{l+1} \binom{r+1}{j}\binom{r}{l-j}\left[\binom{a}{b}\binom{a}{c-2} - \binom{a}{b-1}\binom{a}{c-1}\right] + \sum_{j=0}^{l+1} \binom{r+1}{l+1-j}\binom{r}{j-1}\left[\binom{a}{c-1}\binom{a}{b-1} - \binom{a}{c-2}\binom{a}{b}\right]$$

which equals

$$\sum_{j=0}^{l+1}\left[\binom{r+1}{j}\binom{r}{l-j}-\binom{r+1}{l-j+1}\binom{r}{j-1}\right]\left[\binom{a}{b}\binom{a}{c-2}-\binom{a}{b-1}\binom{a}{c-1}\right].$$

Twice the second of these sums can be expressed as

$$\sum_{j=0}^{l}\binom{r+1}{j}\binom{r}{l-j}\left[\binom{a}{b}\binom{a}{c-1}-\binom{a}{b-1}\binom{a}{c}\right]+\sum_{j=0}^{l}\binom{r+1}{l-j}\binom{r}{j}\left[\binom{a}{b-1}\binom{a}{c}-\binom{a}{b}\binom{a}{c-1}\right]$$

which equals

$$\sum_{j=0}^{l+1}\left[\binom{r+1}{j}\binom{r}{l-j}-\binom{r+1}{l-j}\binom{r}{j}\right]\left[\binom{a}{b}\binom{a}{c-1}-\binom{a}{b-1}\binom{a}{c}\right].$$

Thus, twice the entire coefficient of the α^l term can be expressed as the following new pair of sums:

$$\begin{aligned}2[\alpha]_l = S_1^{(l)}+S_2^{(l)} &= \sum_{j=0}^{l}\left[\binom{r+1}{j}\binom{r}{l-j}-\binom{r+1}{l-j+1}\binom{r}{j-1}\right]\left[\binom{a}{b}\binom{a}{c-2}-\binom{a}{b-1}\binom{a}{c-1}\right]\\ &\quad+\sum_{j=0}^{l}\left[\binom{r+1}{j}\binom{r}{l-j}-\binom{r+1}{l-j}\binom{r}{j}\right]\left[\binom{a}{b}\binom{a}{c-1}-\binom{a}{b-1}\binom{a}{c}\right].\end{aligned}$$

For $l \in \{0, \ldots, 2r\}$, let $T^{(l)} = -S_2^{(l+1)} + S_2^{(l)}$. From the identity

$$\binom{r+1}{j}\binom{r}{l-j}-\binom{r+1}{l-j+1}\binom{r}{j-1}=-\left[\binom{r+1}{j}\binom{r}{l-j+1}-\binom{r+1}{l-j+1}\binom{r}{j}\right],$$

we have

$$\Delta_{r,s}(z)=\left(\frac{1}{2}\right)\left[\sum_{l=0}^{2r}T^{(l)}\alpha^l+(S_1^{(2r+1)}+S_2^{(2r+1)})\alpha^{2r+1}\right].$$

Since $S_1^{(2r+1)} = S_2^{(0)} = 0$, we can rewrite $\Delta_{r,s}(z)$ as

$$\left(\frac{\alpha-1}{2}\right)\sum_{j=0}^{2r}S_2^{(j+1)}\alpha^j$$

which vanishes, is positive, or is negative when $p = q$, $q > p$, or $q < p$, respectively, due to the fact that each of the $S_2^{(l)}$ terms are non-negative. □

3. Example

Consider as an illustrative example the application of statistical inference to random graphs — for instance, social network analysis. Let $G = (V, E)$ be a random graph on the n vertices $\{1, \ldots, n\}$. Assume that the $\binom{n}{2}$ random variables $X_{i,j} = [\text{edge}(i, j) \in E]$ for $i, j \in V$ are independent Bernoulli$(p_{i,j})$. A simplest null hypothesis is *homogeneity* – $p_{i,j} = p \in [0, 1)$ for all $i, j \in V$ (Erdos–Renyi) – and a corresponding alternative hypothesis is that some subset $V_A \subset V$ with $1 < |V_A| \leq n$ has the property that $i, j \in V_A \Rightarrow X_{i,j} \sim$ Bernoulli(q) while all remaining edges are Bernoulli(p), with $q > p$. Assuming that one observes only the size of the graph, $z = |E|$, our MLR result shows that the uniformly most powerful test rejects the null hypothesis for large values of z.

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