Note

On the monotone likelihood ratio property for the convolution of independent binomial random variables

Andrey Rukhin \(^a\), Carey E. Priebe \(^a\,^*\), Dennis M. Healy Jr. \(^b\)

\(^a\) Johns Hopkins University, United States
\(^b\) University of Maryland, United States

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**ABSTRACT**

Given that \( r \) and \( s \) are natural numbers and \( X \sim\) Binomial\((r, q)\) and \( Y \sim\) Binomial\((s, p)\) are independent random variables where \( q, p \in (0, 1) \), we prove that the likelihood ratio of the convolution \( Z = X + Y \) is decreasing, increasing, or constant when \( q < p, q > p \) or \( q = p \), respectively.

**1. Introduction**

Let \( r, s \in \mathbb{N}, z \in \{0, \ldots, r+s\} \), and \( q, p \in (0, 1) \). Let \( X \sim\) Binomial\((r, q)\) and \( Y \sim\) Binomial\((s, p)\) be independent random variables. Let \( P_{r,s}(z) \) denote the likelihood function of the convolution \( Z = X + Y \), so that

\[
P_{r,s}(z) = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{z-k} q^k (1-q)^{r-k} p^{z-k} (1-p)^{s-z+k}.
\]

We will show that the ratio

\[
\frac{P_{r+1,s-1}(z)}{P_{r,s}(z)}
\]

is increasing, decreasing or constant – with respect to \( z \) – when \( q < p, q > p \) or \( q = p \), respectively. Moreover, this result is obtained by only appealing to elementary combinatorial identities.

This same ratio has been analyzed for its monotone likelihood ratio (MLR) properties with respect to fixed \( z \) (Ghurye and Wallace [2], Grayson [3], Huynh [4]). Our main result is that the family of convolutions of independent binomial random variables indexed by parameters \( r, s \) with \( r + s = c \) constant is a MLR family in \( z \).

In statistical inference, MLR families give rise to uniformly most powerful tests — for a given null hypothesis, the same test statistic is known to be optimal (in terms of statistical power) across an entire composite alternative hypothesis (Bickel and Doksum [1], Section 4.3). Our result demonstrates that, for \( r + s = c \) constant and \( q > p \), rejecting \( H_0 : r = 0 \) for large values of the test statistic \( Z \) is most powerful against any alternative \( H_A : r = r' > 0 \).
2. Main result

**Theorem.** The ratio
\[
\frac{P_{r+s-1}(z)}{P_{r,s}(z)}
\]
is increasing, decreasing or constant – with respect to \(z\) – when \(q < p, q > p\) or \(q = p\), respectively.

**Proof.** Fix \(r, s \in \mathbb{N}\) and \(p, q \in (0, 1)\). We are considering the likelihood
\[
P_{r,s}(z) = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{z-k} q^k (1-q)^{r-k} p^{s-k} (1-p)^{r-z+k}
\]
or equivalently
\[
P_{r,s}(z) = [(1-q)^r (1-p)^s] \left( \frac{p}{1-p} \right)^z S_{r,s}(z)
\]
where
\[
S_{r,s}(z) = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{z-k} \alpha^k
\]
and \(\alpha = \frac{q(1-p)}{p(1-q)}\).

In particular, for \(1 \leq z \leq r + s\), we are interested in the difference of the likelihood ratios
\[
\frac{P_{r+s-1}(z)}{P_{r,s}(z)} - \frac{P_{r+s-1}(z-1)}{P_{r,s}(z-1)} = \left( 1 - \frac{q}{1-p} \right) \left( \frac{S_{r+s-1}(z)S_{r,s}(z-1) - S_{r+s-1}(z-1)S_{r,s}(z)}{S_{r,s}(z)S_{r,s}(z-1)} \right).
\]

Let
\[
\Delta_{r,s}(z) = S_{r+s-1}(z)S_{r,s}(z-1) - S_{r+s-1}(z-1)S_{r,s}(z).
\]

We will show that \(\Delta_{r,s}(z)\) vanishes, is positive, or is negative when \(p = q, q > p\), or \(q < p\), respectively.

For legibility, we will use the notation \(a = s - 1\), \(b = z - j\), \(c = z - l + j\), and \(d = z - k\).

First, we will rewrite the quantity \(\Delta_{r,s}(z)\) in terms of powers of \(\alpha\), and apply an elementary combinatorial identity on selected terms:
\[
\Delta_{r,s}(z) = \sum_{j=0}^{r+1} \sum_{k=0}^{r} \binom{r+1}{j} \binom{r}{k} \left[ \binom{a}{b} \binom{a+1}{d-1} - \binom{a}{b-1} \binom{a}{d} \right] \alpha^{j+k}
\]
\[
= \sum_{j=0}^{r+1} \sum_{k=0}^{r} \binom{r+1}{j} \binom{r}{k} \left[ \binom{a}{b} \left( \binom{a}{d-1} + \binom{a}{d-2} \right) - \binom{a}{b-1} \left( \binom{a}{d} + \binom{a}{d-1} \right) \right] \alpha^{j+k}
\]
and thus for each \(l \in \{0, 1, \ldots, 2r + 1\}\) we can express the coefficient of \(\alpha^l\) as
\[
\sum_{j=0}^{r+1} \binom{r+1}{j} \binom{r}{l-j} \left[ \binom{a}{b} \left( \binom{a}{c-1} + \binom{a}{c-2} \right) - \binom{a}{b-1} \left( \binom{a}{c} + \binom{a}{c-1} \right) \right].
\]

We will split this coefficient into a pair of sums
\[
\sum_{j=0}^{l+1} \binom{r+1}{j} \binom{r}{l-j} \left[ \binom{a}{b} \binom{a}{c-2} - \binom{a}{b-1} \binom{a}{c-1} \right]
\]
\[
+ \sum_{j=0}^{l} \binom{r+1}{j} \binom{r}{l-j} \left[ \binom{a}{b} \binom{a}{c-1} - \binom{a}{b-1} \binom{a}{c} \right]
\]
and separately analyze each sum in this pair of sums.

Note that twice the first of these sums can be expressed as
\[
\sum_{j=0}^{l+1} \binom{r+1}{j} \binom{r}{l-j} \left[ \binom{a}{b} \binom{a}{c-2} - \binom{a}{b-1} \binom{a}{c-1} \right]
\]
\[
+ \sum_{j=0}^{l+1} \binom{r+1}{j+1} \binom{r}{l-j} \left[ \binom{a}{c-1} \binom{a}{b-1} - \binom{a}{c-2} \binom{a}{b} \right]
\]
which equals
\[ \sum_{j=0}^{l+1} \left[ \binom{r+1}{j} (r-j) - \binom{r+1}{r-j+1} (j+1) \right] \left[ \binom{a}{b} \binom{a}{c-2} - \binom{a}{b-1} \binom{a}{c-1} \right]. \]

Twice the second of these sums can be expressed as
\[ \sum_{j=0}^{l} \left[ \binom{r+1}{j} (r-j) \right] \left[ \binom{a}{b} \binom{a}{c-1} - \binom{a}{b-1} \binom{a}{c} \right] + \sum_{j=0}^{l} \left[ \binom{r+1}{l-j} \right] \left[ \binom{a}{b-1} \binom{a}{c} - \binom{a}{b} \binom{a}{c-1} \right] \]
which equals
\[ \sum_{j=0}^{l+1} \left[ \binom{r+1}{j} (r-j) - \binom{r+1}{r-j+1} (j+1) \right] \left[ \binom{a}{b} \binom{a}{c-1} - \binom{a}{b-1} \binom{a}{c} \right]. \]

Thus, twice the entire coefficient of the \(\alpha^l\) term can be expressed as the following new pair of sums:
\[ 2|\alpha| l = S_1^{(l)} + S_2^{(l)} = \sum_{j=0}^{l} \left[ \binom{r+1}{j} (r-j) - \binom{r+1}{r-j+1} (j+1) \right] \left[ \binom{a}{b} \binom{a}{c-2} - \binom{a}{b-1} \binom{a}{c-1} \right] + \sum_{j=0}^{l} \left[ \binom{r+1}{l-j} \right] \left[ \binom{a}{b-1} \binom{a}{c} - \binom{a}{b} \binom{a}{c-1} \right]. \]

For \(l \in \{0, \ldots, 2r\}\), let \(T^{(l)} = -S_2^{(l+1)} + S_2^{(l)}\). From the identity
\[ \binom{r+1}{j} (r-j) - \binom{r+1}{r-j+1} (j+1) = -\left[ \binom{r+1}{j} (r-j) - \binom{r+1}{r-j+1} (j+1) \right], \]
we have
\[ \Delta_{r,s}(z) = \left( \frac{1}{2} \right) \left[ \sum_{l=0}^{2r} T^{(l)} \alpha^l + (S_1^{(2r+1)} + S_2^{(2r+1)}) \alpha^{2r+1} \right]. \]

Since \(S_1^{(2r+1)} = S_2^{(0)} = 0\), we can rewrite \(\Delta_{r,s}(z)\) as
\[ \left( \frac{\alpha - 1}{2} \right) \sum_{l=0}^{2r} S_2^{(l+1)} \alpha^l \]
which vanishes, is positive, or is negative when \(p = q, q > p\), or \(q < p\), respectively, due to the fact that each of the \(S_2^{(l)}\) terms are non-negative. \(\square\)

3. Example

Consider an illustrative example the application of statistical inference to random graphs — for instance, social network analysis. Let \(G = (V, E)\) be a random graph on the \(n\) vertices \(\{1, \ldots, n\}\). Assume that the \(\binom{n}{2}\) random variables \(X_{ij} = \text{edge}(i, j) \in E\) for \(i, j \in V\) are independent Bernoulli(\(p_{ij}\)). A simplest null hypothesis is homogeneity - \(p_{ij} = p \in [0, 1]\) for all \(i, j \in V\) (Erdos–Renyi) — and a corresponding alternative hypothesis is that some subset \(V_A \subset V\) with \(1 < |V_A| \leq n\) has the property that \(i, j \in V_A \Rightarrow X_{ij} \sim \text{Bernoulli}(q)\) while all remaining edges are Bernoulli(\(p\)), with \(q > p\). Assuming that one observes only the size of the graph, \(z = |E|\), our MLR result shows that the uniformly most powerful test rejects the null hypothesis for large values of \(z\).

References