Infinite dimensional sub-Riemannian geometry

Sylvain Arguillère
(CIS, Johns Hopkins University)

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We will see that for sub-Riemannian Banach manifolds, even the strong case presents several significant difficulties preventing the generalization of certain finite dimensional and/or Riemannian results. In particular, there is no Pontryagin maximum principle, hence some geodesics can be neither normal nor abnormal.

Plan

Relative tangent spaces

Let M be a smooth manifold. Usually, a sub-Riemannian structure on M is a couple $(\mathcal{H}, g)$, where $\mathcal{H}$ is a sub-bundle of TM (i.e. a *distribution* on M) and g a Riemannian metric on $\mathcal{H}$. However, this does not take into account rank-varying structures.

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Definition

A rank-varying distribution of subspaces on a Banach manifold M of class $\mathcal{C}^k$, also called a **relative tangent space** *of class $\mathcal{C}^k$, is a couple $(\mathcal{H}, \xi)$, where $\mathcal{H}$ is a smooth Banach vector bundle on M and $\xi : \mathcal{H} \to TM$ is a vector bundle morphism of class $\mathcal{C}^k$.*

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In other words, for $q \in M$, ξ_q is a linear mapping $\mathcal{H}_q \to T_qM$.

The distribution of subspaces $\xi(\mathcal{H}) \subset TM$ is called the *horizontal bundle*.

Définition

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A **sub-Riemannian structure** *on a Banach manifold M is a triple $(\mathcal{H}, \xi, g)$, with $(\mathcal{H}, \xi)$ a relative tangent bundle on M and g a smooth pre-Hilbertian metric on $\mathcal{H}$.*

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The structure is called **strong** when g defines a Hilbert norm on each fiber, and **weak** in all other cases.

Horizontal vector fields and curves

Horizontal vector fields and curves are those that are everywhere tangent to $\xi(\mathcal{H})$, i.e.:

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$$\exists u \in \Gamma(\mathcal{H}), \quad X(q) = \xi_q u(q), \quad q \in M.$$

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A curve $t \mapsto q(t)$ of Sobolev class H^1 is **horizontal** *if*

$$\exists t \mapsto u(t) \in \mathcal{H}_{q(t)}, \quad \dot{q}(t) = \xi_{q(t)} u(t)$$

for almost every t, with $(q(\cdot), u(\cdot)) \in L^2$. The couple $(q(\cdot), u(\cdot))$ is a **horizontal system***, q is the* **trajectory** *while u is the* **control***.*

Example: the Heisenberg group

$M = \mathbb{R}^3$, and horizontal distribution generated by

$$X_1 = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}.$$

- Horizontal vector field: $X = uX_1 + vX_2$, $u, v : \mathbb{R}^3 \to \mathbb{R}^3$.
- Horizontal curve: $\dot{z}(t) = \frac{1}{2}(x\dot{y} - \dot{x}y), \dot{x}, \dot{y} \in L^2(0, 1; \mathbb{R})$.

Length and action

Length and action of a horizontal system (q, u):

$$L(q, u) = \int \sqrt{g_{q(t)}(u(t), u(t))}dt,$$

$$A(q, u) = \frac{1}{2}\int g_{q(t)}(u(t), u(t))dt.$$

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Length of a curve $q(\cdot)$:

$$L(q) = \inf_{u,(q,u) \text{ horizontal}} L(q,u).$$

For strong structures, there always exists a unique u such that $L(q,u) = L(q)$. However, easier to work on systems than on curves.

Sub-riemannian distance

- **Sub-riemannian distance:**

$$\forall q_0, q_1 \in M, d(q_0, q_1) = \inf_{\substack{(q,u)\in L^2(0,1;\mathcal{H}),\\ (q,u) \text{ horizontal},\\ q(0)=q_0,\ q(1)=q_1}} L(q, u).$$

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- **Orbit of** $q_0 \in M$: $\mathcal{O}_{q_0} = \{q \mid d(q_0, q) < +\infty\}$.
- **Geodesic:** curve q for which there exists a control u such that (q, u) minimizes the action A with fixed endpoints on small enough intervalls.

Plan

Lie algebra generated by the structure

Let $\mathcal{L} \subset \Gamma(TM)$ be the Lie algebra of vector fields generated by smooth and horizontal vector fields. We can identify it with a subset of TM using $\mathcal{L}_q = \{X(q) \mid X \in \mathcal{L}\}$.

Theorem (Chow-Rashevski)

For M connected and finite dimensional, if $\mathcal{L} = TM$, then the distance between any two points in M is finite **(controlability)***. Moreover, the topology induced by d coincides with its intrisic manifold topology.*

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Open problem when $\dim(M) = \infty$. Moreover, $\mathcal{L}_q$ is rarely even a closed subset of T_qM, because $\mathcal{L}$ is generated algebraically.

Example: product of Heisenberg groups

$$M = \ell^2(\mathbb{N}, \mathbb{R}^3) = \left\{ (x_n, y_n, z_n)_{n\in\mathbb{N}} \mid \sum_{n=0}^{\infty} x_n^2 + y_n^2 + z_n^2 < +\infty \right\}.$$

Horizontal curves:

$$\dot{z}_n = \frac{1}{2}(x_n \dot{y}_n - \dot{x}_n y_n), \quad n \in \mathbb{N}.$$

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Horizontal curves:

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Then

$$d(0, (x_n, y_n, z_n))^2 \simeq \sum_{n=0}^{\infty} x_n^2 + y_n^2 + |z_n|.$$

Hence $\mathcal{O}_0 = \ell^2(\mathbb{N}; \mathbb{R}^2) \times \ell^1(\mathbb{N}; \mathbb{R})$ is only dense with empty interior in M: **approximate controllability**.

Conditions for approximate controllability

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The proof makes use of the following lemma.

Lemma

Assume M connected. Let $B \subset M$ be a closed subset such that T_qB (velocities at q of curves in B) is dense in T_qM. Then $B = M$.

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Recently generalized to **convenient spaces** by Grong, Markina et Vasil'ev.

Conditions for exact controllability

Let $X_1, \ldots, X_r$ be smooth horizontal vector fields. For $I = (i_1, \ldots, i_k)$ define

$$X_I = [X_{i_k}, [X_{i_{k-1}}, \ldots, [X_{i_2}, X_{i_1}] \ldots].$$

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We say that the structure satisfies the **strong Chow-Rashevski property** at $q \in M$ if,

$$\exists n \in \mathbb{N},\ \forall Z(q) \in T_qM, \quad Z(q) = \sum_{k=0}^{n} \sum_{I \in \{1,\ldots,r\}^k} [Y_I, X_I](q)$$

with each Y_I a horizontal vector field.

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with each Y_I a horizontal vector field.

Example: $M = \mathbb{R} \times \ell^2(\mathbb{N}, \mathbb{R}^2)$, and horizontal vector fields generated by

$$X(x, y_n, z_n) = \frac{\partial}{\partial x}, \quad Y_n(x, y_n, z_n) = \frac{\partial}{\partial y_n} + x\frac{\partial}{\partial z_n}.$$

Conditions for exact controllability

When this property is satisfied, one can adapt the finite dimensional proof and obtain:

Theorem

Assume M is connected and the structure satisfies the strong Chow-Rashevski theorem at every $x \in M$. Then M is an orbit, and the topology induced by the sub-Riemannian distance is coarser than the intrisic manifold topology (for strong structure, the topologies coincide, and M is Hilbert).

Plan

1. Infinite dimensional sub-Riemannian geometry
   - Definitions and examples
   - Accessibility and the Chow-Rashevski theorem
   - Hamiltonians and geodesic flow
2. Right-invariant sub-Riemannian geometry on a group of diffeomorphisms
   - Definition and first results
   - Chow-Rashevski on $\mathcal{D}^s(M)$
   - Reproducing kernel and the geodesic flow

Set of horizontal systems

Let $q_0 \in M$ be fixed. Define

$$\Omega_{q_0}^{\mathcal{H}} = \{(q,u) \in L^2(0,1;\mathcal{H}) \mid (q,u) \text{ horizontal } q(0)=q_0\}$$

To equip $\Omega_{q_0}^{\mathcal{H}}$ with a manifold structure, we would need a local addition on M (exists for Lie groups for example).

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However, on $U \subset M$ open such that $\mathcal{H}_{|U} \simeq U \times H$, one can identify horizontal systems in $\mathcal{H}_{|U}$ with an open subset of $L^2(0,1;H)$.

Endpoint mapping

Definition

Soit $q_0 \in M$. The **endpoint mapping** $\mathrm{end} : \Omega^{\mathcal{H}}_{q_0} \to M$ *is given by* $\mathrm{end}(q, u) = q(1)$. *Its regularity is the same as that of ξ.*

We have $\mathcal{O}_{q_0} = \mathrm{end}(\Omega^{\mathcal{H}}_{q_0})$.

Constrained optimisation

Lemma

Let $\mathcal{M}$ be a Banach manifold, $F : \mathcal{M} \to \mathbb{R}$ a smooth cost and $C : \mathcal{M} \to M$ smooth constraints. If $u \in \mathcal{M}$ minimizes F on $C^{-1}(q)$, then $d(F, C)(u) = (dF(u), dC(u))$ is not surjective.

If $dim(M) < \infty$, the conclusion is equivalent to

$$\exists \lambda \in \{0, 1\}, p \in T_q^* M, \quad \lambda dF(u) = dC(u)^* p,$$

with λ or p non-zero. However, if $dim(M) = \infty$, the image of $d(F, C)(u)$ may be dense.

Three kinds of geodesics

We apply this lemma with $F = A$ and $C = \mathrm{end}$.

Lemma

Let $(q, u) \in \Omega_{q_0}^{\mathcal{H}}$ be a minimizing geodesic connecting q_0 and $\mathrm{end}(q, u) = q_1$. Then one of the following statements is true:

1. $\exists(\lambda, p_1) \in \{0, 1\} \times T_{q_1}^* M \setminus \{(0, 0)\}$,

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If $\lambda = 1$: **normal geodesic**.
If $\lambda = 0$, (q, u) is a singular point of end: **abnormal geodesic**.

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1. $\exists (\lambda, p_1) \in \{0,1\} \times T^*_{q_1}M \setminus \{(0,0)\}$,

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2. *$Im(dA(q,u), d\,\mathrm{end}(q,u))$ is a proper dense subset of $\mathbb{R} \times T_{q_1}M$ (**elusive geodesic**).*

In the end, this tells us nothing.

Partial converse

Lemma

Let $\mathcal{M}$ be a Banach manifold, $F : \mathcal{M} \to \mathbb{R}$ a smooth cost and $C : \mathcal{M} \to M$ smooth constraints. Let $u \in \mathcal{M}$. Then

1. *If there exists $p \in T_q^*M$ such that $dF(u) = dC(u)^*p$, then u is a critical point of F on $C^{-1}(q)$.*
2. *If there exists $0 \neq p \in T_q^*M$ such that $0 = dC(u)^*p$, then u is a critical point of C.*

Converse

Let us apply our partial converse.

Lemma

Let $(q, u) \in \Omega_{q_0}^{\mathcal{H}}$ *with* $q(1) = q_1$ *and* $p_1 \in T_{q_1}^* M$.

1. *If* $dA(q, u) = d\,\mathrm{end}(q, u)^* p_1$ *then* (q, u) *is a critical point of the action with fixed endpoints.*
2. *If* $p_1 \neq 0$ *and* $0 = d\,\mathrm{end}(q, u)^* p_1$*, then* (q, u) *is a critical point of* end*. We say that* q *is a* **singular curve**.

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2. *If* $p_1 \neq 0$ *and* $0 = d\,\mathrm{end}(q, u)^* p_1$, *then* (q, u) *is a critical point of* end. *We say that* q *is a* **singular curve**.

Obviously, not every critical point of A with fixed endpoints have this form.

Hamiltonian

Let ω be the canonical weak symplectic form of T^*M, and $\lambda \in \{0,1\}$. Define $H^\lambda : T^*M \oplus \mathcal{H} \to \mathbb{R}$:

$$\boxed{H^\lambda(q,p,u) = (p \mid \xi_q u) - \frac{\lambda}{2} g_q(u,u).}$$

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If $\partial_u H^\lambda(q,p,u) = 0$, then

$$\nabla^\omega H^\lambda(q,p,u) = (\partial_p H, -\partial_q H) \in T_{(q,p)} T^*M$$

is defined intrinsically (since $\partial_p H(q,p,u) \in T_q^{**}M$ can be identified to $\xi_q u \in T_q M$).

Hamiltonian formulation

Proposition

$$\lambda dA(q,u) = d\,\mathrm{end}(q,u)^* p_1 \iff \exists t \mapsto p(t) \in T^*_{q(t)}M,\ \ p(1) = p_1,$$

$$\begin{cases} \quad 0 = \partial_u H^\lambda(q(t), p(t), u(t)), \\ (\dot q(t), \dot p(t)) = \nabla^\omega H^\lambda(q(t), p(t), u(t)) = (\partial_p H^\lambda, -\partial_q H^\lambda). \end{cases}$$

Normal Hamiltonian

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$$\partial_u H^1(q,p,u) = 0 \iff g_q(u,\cdot) = \xi_q^* p \in \mathcal{H}_q^*$$

$$\Leftrightarrow \boxed{H^1(q,p,u) = \max_{u' \in \mathcal{H}_q} H^1(q,p,u') := h(q,p).}$$

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The mapping $h : T^*M \to \mathbb{R} \cup \{+\infty\}$ is the **normal Hamiltonian**.

Let us start with the strong case.

Let $K_q : \mathcal{H}_q^* \to \mathcal{H}_q$ such that $\forall \alpha \in \mathcal{H}_q^*,\ \alpha = g_q(K_q\alpha, \cdot)$. K is a smooth vector bundle morphism.

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$$g_q(u, \cdot) = \xi_q^* p \quad \Longleftrightarrow \quad u = K_q \xi_q^* p,$$

so that $h(q,p) = \frac{1}{2} p\xi_q K_q \xi_q^* p$ is as smooth as ξ. Then $\partial_p h(q,p) \in T_q^{**}M$ can be identified to $\xi_q K_q \xi_q^* p \in T_qM$.

Consequence: h has a symplectic gradient, which is as smooth as $d\xi$.

Geodesic flow

Theorem

For a strong structure with ξ of class $\mathcal{C}^3$, h admits a symplectic gradient of classe $\mathcal{C}^2$, and for any initial condition $(q(0), p(0)) \in T^*M$, there exists a unique solution $t \mapsto (q(t), p(t)) \in T^*M$ such that

$$\begin{aligned}(\dot{q}(t), \dot{p}(t)) &= \nabla^\omega h(q(t), p(t)) \\ &= (\partial_p h(q, p), -\partial_q h(q, p)).\end{aligned}$$

Then $q(\cdot)$ **is indeed a (normal) geodesic.**

Elusive geodesics

Remark: By restricting the structure to a stable proper dense $M' \subset M$ submanifold, we obtain bigger cotangent spaces and hence "more" normal geodesics. Those geodesics were elusive in M.

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Open problem: is there a "right" cotangent space (such that elusive geodesics disappear).

Example: on the infinite product of Heisenberg groups, as $\mathcal{O}_0 = \ell^2(\mathbb{N}, \mathbb{R}^2) \times \ell^1(\mathbb{N}, \mathbb{R})$, taking $p(0)$ in $\ell^2(\mathbb{N}, \mathbb{R}^2) \times \ell^\infty(\mathbb{N}, \mathbb{R})$ makes all minimizing geodesics into normal geodesics.

The weak case and adapted cotangent sub-bundles

Difficulty: for weak structures, $h(q, p)$ may be infinite.

Definition

A **dense** *sub-bundle* $\tau^*M \hookrightarrow T^*M$ *is said to be* **adapted** *to the structure if the restriction of h to* τ^*M *is finite.*

The restriction of ω to such a sub-bundle remains a weak symplectic form. However, the restriction of h has a symplectic gradient only when, in a local trivialization, we can write $\partial_q h(q, p) \in \tau_q^* M$.

Example: for the Riemannian case $\mathcal{H} = TM$, simply take $\tau^* M = g(TM, \cdot)$.

Geodesic flow

Theorem (page 39)

Let τ^*M be an adapted cotangent sub-bundle on which h possesses a $\mathcal{C}^2$ symplectic gradient. Then for any initial condition $(q(0), p(0)) \in \tau^*M$, there exists a unique solution $t \mapsto (q(t), p(t)) \in \tau^*M$ such that

$$\begin{aligned}(\dot{q}(t), \dot{p}(t)) &= \nabla^\omega h(q(t), p(t)) \\ &= (\partial_p h(q,p), -\partial_q h(q,p)).\end{aligned}$$

In this case, $q(\cdot)$ is a local geodesic.

Plan

The group of H^s diffeomorphisms

Let M be a complete manifold of bounded geometry. This lets us define the space $\Gamma^s(TM)$ of vector fields of class H^s, $s > dim(M)/2 + 1$.

Let $\mathcal{D}^s(M) = \exp(\Gamma^s(TM)) \cap \mathrm{Diff}^1(M)$.

This space is a Hilbert manifold and a topological group for the composition law, with

$$T_\varphi \mathcal{D}^s(M) = \Gamma^s(TM) \circ \varphi.$$

Right-invariant vector fields

The mapping $\varphi \mapsto \varphi \circ \psi$ is smooth for every ψ.

While $\varphi \mapsto \psi \circ \varphi$ is of class $\mathcal{C}^k$ whenever ψ is actually of class H^{s+k}.

So if $X \in \Gamma^{s+k}(TM)$, then $\varphi \in \mathcal{D}^s(M) \mapsto X \circ \varphi \in T_\varphi \mathcal{D}^s(M)$ is of class $\mathcal{C}^k$.

Consequence: If $t \mapsto \varphi(t)$ is a $\mathcal{C}^1$-curve $\mathcal{D}^s(M)$ starting at Id_M, then $\varphi(t)$ is the flow of the time dependent vector field $X(t) = \dot{\varphi}(t) \circ \varphi(t)^{-1}$, that is,

$$\dot{\varphi}(t) = X(t) \circ \varphi(t).$$

Moreover, $t \mapsto X(t)$ is continuous in time.

Right-invariant SR structures

Let $(V, \langle\cdot,\cdot\rangle)$ be a Hilbert space of vector fields with continuous inclusion in $\Gamma^{s+k}(TM)$.

The mapping $(\varphi, X) \mapsto X \circ \varphi$ from $\mathcal{D}^s(M) \times V$ into $T\mathcal{D}^s(M)$ defines a $\mathcal{C}^k$ relative tangent space, which, in addition to the Hilbert product $\langle\cdot,\cdot\rangle$, then *defines a strong sub-Riemannian structure* on $\mathcal{D}^s(M)$.

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Horizontal curves $t \mapsto \varphi(t)$ are those such that there exists $X \in L^2(0, 1; V)$ such that, almost everywhere,

$$\dot{\varphi}(t) = X(t) \circ \varphi(t).$$

So they are just flows of time-dependent vector fields of V. The energy is

$$\frac{1}{2} \int_0^1 \langle X(t), X(t) \rangle \, dt.$$

Sub-Riemannian distance

We also define length, sub-Riemannian distance, and geodesics as usual.

Proposition

The sub-Riemannian distance is right-invariant, complete, and any two diffeomorphisms with finite distance from one another can be connected by a minimizing geodesic.

Moreover, $\mathcal{O}_{\mathrm{Id}_M} = \{\varphi \in \mathcal{D}^s(M) \mid d(\mathrm{Id}_M, \varphi) < \infty\}$ *is a subgroup of* $\mathcal{D}^s(M)$.

Plan

Horizontal flows

Let $X_1, \dots, X_r$ be smooth vector fields on M satisfying the Chow-Rashevski bracket generating condition. This defines a sub-Riemannian structure on M.

Assume that V is the set of vector fields X of the form

$$X(x) = \sum_{i=1}^{k} u^i(x) X_i(x), \quad u^i \in H^s(M),$$

that is, the set of horizontal vector fields of class H^s.

This means that $t \mapsto \varphi(t)$ is horizontal if and only if each curve $t \mapsto \varphi(t,x)$ is horizontal on M.

Accessible set

Theorem

If M is compact, then $G^V = \mathcal{D}_0^s(M)$. Moreover, the topology induced by the sub-Riemannian distance coincides with the manifold topology.

In other words, if any two points on M can be connected by a horizontal curve, any two diffeomorphisms of M can be connected by composition with the flow of a horizontal vector field.

Remark: Not true when M is not compact.

Plan

Hamiltonian for $M = \mathbb{R}^d$

On $\mathbb{R}^d$, $\mathcal{D}^s(\mathbb{R}^d)$ is the set of diffeomorphisms of the form $\varphi(x) = x + X(x)$, with $X \in H^s(\mathbb{R}^d, \mathbb{R}^d)$. So we can just write $T_\varphi \mathcal{D}^s(\mathbb{R}^d) = H^s(\mathbb{R}^d, \mathbb{R}^d)$.

Hence, $T^*_\varphi \mathcal{D}^s(\mathbb{R}^d) = H^{-s}(\mathbb{R}^d, (\mathbb{R}^d)^*)$, that is, the set of distributional valued 1-forms with coefficients in H^{-s}, and

$$(p \mid X) = \int_{\mathbb{R}^d} p(x) X(x) dx.$$

The Hamiltonian $H : T^*\mathcal{D}^s(\mathbb{R}^d) \to \mathbb{R}$ is

$$H(\varphi, p) = \max_{X \in V} \int_{\mathbb{R}^d} p(x) X(\varphi(x)) dx - \frac{1}{2} \langle X, X \rangle .$$

Reproducing kernel

$V \hookrightarrow \mathcal{C}^0(\mathbb{R}^d, \mathbb{R}^d)$ implies that $V^* \to V$ is obtained by convolution with a kernel $(x, y) \mapsto K(x, y) \in \mathbb{R}^d \otimes \mathbb{R}^d$.

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Let $p \in H^s(\mathbb{R}^d, \mathbb{R}^d)^* = \{$1-forms on $\mathbb{R}^d$with coefficients in $H^{-s}\} \subset V^*$. then

$$p_{|V} = \langle X, \cdot \rangle \;\Leftrightarrow\; X(x) = \int_M K(x, y)p(y)dy, \quad x \in M.$$

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Conversely, any symmetric positive definite convolution kernel K defines a unique Hilbert space of vector fields V.

Hamiltonian for $M = \mathbb{R}^d$

We can compute the Hamiltonian thanks to the kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}((\mathbb{R}^d)^*, \mathbb{R}^d)$:

$$H(\varphi, p) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} p(x) K(\varphi(x), \varphi(y)) p(y) dy dx.$$

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When $V \hookrightarrow H^{s+2}(\mathbb{R}^d, \mathbb{R}^d)$, the Hamiltonian is $\mathcal{C}^2$, and the Hamiltonian equations

$$\dot{\varphi}(t) = \partial_p H(\varphi(t), p(t)), \quad \dot{p}(t) = -\partial_\varphi H(\varphi(t), p(t))$$

have a unique solution for fixed $(\varphi_0, p_0) \in T^*\mathcal{D}^s(M)$.

Hamiltonian equations

Theorem

Let $t \mapsto (\varphi(t), p(t))$ satisfy the geodesic equations

$$\partial_t \varphi(t,x) = \int_{\mathbb{R}^d} K(\varphi(t,x), \varphi(t,y)) p(t,y) dy$$

and

$$\partial_t p(t,x) = -p(t,x) \int_{\mathbb{R}^d} \partial_1 K(\varphi(t,x), \varphi(t,y)) p(t,y) dy.$$

Then, if H is of class $\mathcal{C}^3$, φ is a geodesic on small enough intervals.

H is of class $\mathcal{C}^3$, for example, when we have a continuous inclusion of V in $H^{s+3}(\mathbb{R}^d, \mathbb{R}^d)$, but not only.

Remarks

A lot of properties of p are preserved along the Hamiltonian equations. In particular, the support of p is constant. This is well-known, for example in landmarks: momentum can only be exchanged between points that already had momentum to begin with.

The (negative) Sobolev regularity of p as a distribution is also preserved.