Sub-Riemannian geometry in groups of diffeomorphisms and shape spaces

Sylvain Arguillère, Emmanuel Trélat (Paris 6),
Alain Trouvé (ENS Cachan), Laurent Younès (JHU)

May 2013

Plan

Plan

Relative tangent spaces

Let M be a smooth manifold. Usually, sub-Riemannian structure on M is a couple $(\mathcal{H}, g)$, where $\mathcal{H}$ is a sub-bundle of TM (i.e. a *distribution* on M) and g a Riemannian metric on $\mathcal{H}$. Not general enough for shape spaces: we need rank-varying distributions.

Definition 1

(Agrachev et al.) *A rank-varying distribution of subspaces on M of class $\mathcal{C}^k$, also called a* **relative tangent space** *of class $\mathcal{C}^k$, is a couple $(\mathcal{H}, \xi)$, where $\mathcal{H}$ is a smooth vector bundle on M and $\xi : \mathcal{H} \to TM$ is a vector bundle morphism of class $\mathcal{C}^k$.*

In other words, for $x \in M$, ξ_x is a linear map $\mathcal{H}_x \to T_x M$.

The distribution of subspaces $\xi(\mathcal{H}) \subset TM$ is called the *horizontal bundle*.

Sub-Riemannian structures

Definition 2

A sub-Riemannian structure on M is a triplet $(\mathcal{H}, \xi, g)$, where $(\mathcal{H}, \xi)$ is a relative tangent space and g a Riemannian metric on $\mathcal{H}$.

A vector field X on M is **horizontal** *if $X = \xi e$, for some section e of $\mathcal{H}$, i.e. if it is tangent to the horizontal distribution.*

A curve $q : [0, 1] \to M$ with square-integrable velocity is horizontal if it is tangent to the distribution, that is, if there exists $u \in L^2(0, 1; \mathcal{H})$ with $u(t) \in \mathcal{H}_{q(t)}$ such that $\dot{q}(t) = \xi_{q(t)} u(t)$.

Its energy is defined by $\frac{1}{2} \int_0^1 g_{q(t)}(u(t), u(t))dt$. The sub-Riemannian length, distance and geodesics are defined just as in the Riemannian case.

Locally, we can take an orthonormal frame $e_1, \dots, e_k$ of $\mathcal{H}$ and define the vector fields $X_i(x) = \xi_x e_i(x)$, so that horizontal curves satisfy

$$\dot{q}(t) = \sum_{i=1}^{k} u^i(t) X_i(q(t)), \quad u(t) = \sum_{i=1}^{k} u_i(t) e_i(q(t)).$$

Its energy is given by $\frac{1}{2} \sum_{i=1}^{k} \int_0^1 u^i(t)^2 dt$.

Horizontal vector fields satisfy

$$X(x) = \sum_{i=1}^{k} u_i(x) X_i(x), \quad x \in M, \quad u_i : M \to \mathbb{R}.$$

Accessibility

A Riemannian structure satisfies the Chow-Rashevski condition if any tangent vector on M is a linear combination of iterated Lie brackets of horizontal vector fields.

Theorem 1

(Chow-Rashevski) *In this case, any two points in M can be joined by a horizontal geodesic. Moreover, the topology defined by the sub-Riemannian distance coincides with its intrinsic manifold topology.*

Hamiltonian

The Hamiltonian of the structure $H : T^*M \to \mathbb{R}$ is defined by

$$H(x,p) = \sup_{u \in \mathcal{H}_x} p(\xi_x u) - \frac{1}{2} g_x(u,u) = \sup_{u \in \mathbb{R}^k} \sum_{i=1}^{k} u^i p(X_i(x)) - \frac{1}{2}(u^i)^2.$$

Note that the $X_i(x)$ are linearly independent, then

$$H(x,p) = \frac{1}{2} \sum_{i=1}^{k} p(X_i(x))^2.$$

Geodesic equations

Theorem 1

Let $(q, p) : [0, 1] \to T^*M$ satisfying the Hamiltonian equations

$$\dot{q}(t) = \partial_p H(q(t), p(t)), \quad \dot{p}(t) = -\partial_q H(q(t), p(t)).$$

Then q is a geodesic on small enough intervals.

Contrarily to the Riemannian case, the converse is **not true**.

Plan

The group of H^s diffeomorphisms

Let M be a complete manifold of bounded geometry. This lets us define the space $\Gamma^s(TM)$ of vector fields of class H^s, $s > dim(M)/2 + 1$.

Let $\mathcal{D}^s(M) = \exp(\Gamma^s(TM)) \cap \mathrm{Diff}^1(M)$.

This space is a Hilbert manifold and a topological group for the composition law, with

$$T_\varphi \mathcal{D}^s(M) = \Gamma^s(TM) \circ \varphi.$$

Right-invariant vector fields

The mapping $\varphi \mapsto \varphi \circ \psi$ is smooth for every ψ.

While $\varphi \mapsto \psi \circ \varphi$ is of class $\mathcal{C}^k$ whenever ψ is actually of class H^{s+k}.

So if $X \in \Gamma^{s+k}(TM)$, then $\varphi \in \mathcal{D}^s(M) \mapsto X \circ \varphi \in T_\varphi \mathcal{D}^s(M)$ is of class $\mathcal{C}^k$.

Consequence: If $t \mapsto \varphi(t)$ is a $\mathcal{C}^1$-curve $\mathcal{D}^s(M)$ starting at Id_M, then $\varphi(t)$ is the flow of the time dependent vector field $X(t) = \dot{\varphi}(t) \circ \varphi(t)^{-1}$, that is,

$$\dot{\varphi}(t) = X(t) \circ \varphi(t).$$

Moreover, $t \mapsto X(t)$ is continuous in time.

Right-invariant SR structures

Let $(V, \langle\cdot,\cdot\rangle)$ be a Hilbert space of vector fields with continuous inclusion in $\Gamma^{s+k}(TM)$.

The mapping $(\varphi, X) \mapsto X \circ \varphi$ from $\mathcal{D}^s(M) \times V$ into $T\mathcal{D}^s(M)$ defines a $\mathcal{C}^k$ relative tangent space, which, in addition to the Hilbert product $\langle\cdot,\cdot\rangle$, then *defines a strong sub-Riemannian structure* on $\mathcal{D}^s(M)$.

Horizontal curves $t \mapsto \varphi(t)$ are those such that there exists $X \in L^2(0,1;V)$ such that, almost everywhere,

$$\dot\varphi(t) = X(t) \circ \varphi(t).$$

So they are just flows of time-dependent vector fields of V. The energy is

$$\frac{1}{2}\int_0^1 \langle X(t), X(t)\rangle \, dt.$$

Sub-Riemannian distance

We also define length, sub-Riemannian distance, and geodesics as usual.

Proposition 1

The sub-Riemannian distance is right-invariant, complete, and any two diffeomorphisms with finite distance from one another can be connected by a minimizing geodesic.

Moreover, $G^V = \{\varphi \in \mathcal{D}^s(M) \mid d(\mathrm{Id}_M, \varphi) \leq \infty\}$ *is a subgroup of* $\mathcal{D}^s(M)$.

Remark: Almost every infinite dimensional LDDMM methods are actually sub-Riemannian, not Riemannian. This does not make the methods wrong, because the control theoritic/Hamiltonian point of view are used, which do not depend on a Riemannian setting.

Horizontal flows

Let $X_1, \ldots, X_r$ be smooth vector fields on M satisfying the Chow-Rashevski bracket generating condition. This defines a sub-Riemannian structure on M.

Assume that V is the set of vector fields X of the form

$$X(x) = \sum_{i=1}^{k} u^i(x) X_i(x), \quad u^i \in H^s(M),$$

that is, the set of horizontal vector fields of class H^s.

This means that $t \mapsto \varphi(t)$ is horizontal if and only if each curve $t \mapsto \varphi(t, x)$ is horizontal on M.

Accessible set

Theorem 2

(No full proof yet) If M is compact, then $G^V = \mathcal{D}_0^s(M)$. Moreover, the topology induced by the sub-Riemannian distance coincides with the manifold topology.

In other words, if any two points on M can be connected by a horizontal curve, any two diffeomorphisms of M can be connected by composition with the flow of a horizontal vector field.

This is very rare in infinite dimensional sub-Riemannian geometry.

Remark: Not true when M is not compact.

Hamiltonian for $M = \mathbb{R}^d$

On $\mathbb{R}^d$, $\mathcal{D}^s(\mathbb{R}^d)$ is the set of diffeomorphisms of the form $\varphi(x) = x + X(x)$, with $X \in H^s(\mathbb{R}^d, \mathbb{R}^d)$. So we can just write $T_\varphi \mathcal{D}^s(\mathbb{R}^d) = H^s(\mathbb{R}^d, \mathbb{R}^d)$.

Hence, $T_\varphi^* \mathcal{D}^s(\mathbb{R}^d) = H^{-s}(\mathbb{R}^d, (\mathbb{R}^d)^*)$, that is, the set of distributional valued 1-forms with coefficients in H^{-s}, and

$$(p \mid X) = \int_{\mathbb{R}^d} p(x) X(x) dx.$$

The Hamiltonian $H : T^*\mathcal{D}^s(\mathbb{R}^d) \to \mathbb{R}$ is

$$H(\varphi, p) = \max_{X \in V} \int_{\mathbb{R}^d} p(x) X(\varphi(x)) dx - \frac{1}{2} \langle X, X \rangle .$$

Hamiltonian for $M = \mathbb{R}^d$

We can compute the Hamiltonian thanks to the kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}((\mathbb{R}^d)^*, \mathbb{R}^d)$:

$$H(\varphi, p) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} p(x) K(\varphi(x), \varphi(y)) p(y) dy dx.$$

When $V \hookrightarrow H^{s+2}(\mathbb{R}^d, \mathbb{R}^d)$, the Hamiltonian is $\mathcal{C}^2$, and the Hamiltonian equations

$$\dot{\varphi}(t) = \partial_p H(\varphi(t), p(t)), \quad \dot{p}(t) = -\partial_\varphi H(\varphi(t), p(t))$$

have a unique solution for fixed $(\varphi_0, p_0) \in T^*\mathcal{D}^s(M)$.

Hamiltonian equations

Theorem 3

Let $t \mapsto (\varphi(t), p(t))$ satisfy the geodesic equations

$$\partial_t \varphi(t,x) = \int_{\mathbb{R}^d} K(\varphi(t,x), \varphi(t,y)) p(t,y) dy$$

and

$$\partial_t p(t,x) = -p(t,x) \int_{\mathbb{R}^d} \partial_1 K(\varphi(t,x), \varphi(t,y)) p(t,y) dy.$$

Then, if H is of class $\mathcal{C}^3$, φ is a geodesic on small enough intervals.

H is of class $\mathcal{C}^3$, for example, when we have a continuous inclusion of V in $H^{s+3}(\mathbb{R}^d, \mathbb{R}^d)$, but not only.

Remarks

A lot of properties of p are preserved along the Hamiltonian equations. In particular, the support of p is constant. This is well-known, for example in landmarks: momentum can only be exchanged between points that already had momentum to begin with.

The (negative) Sobolev regularity of p as a distribution is also preserved.

Example: sub-Riemannian Gaussian kernels

On $\mathbb{R}^d$, let $X_1, \dots, X_r$ be smooth vector fields of polynomial growth. Define

$$K(x,y)p = e^{-\frac{|x-y|^2}{2\sigma}} \sum_{i=1}^{r} p(X_i(y))X_i(x).$$

Then $X \in V$ is horizontal for the sub-Riemannian structure on M induced by the X_is. The Hamiltonian becomes

$$H(\varphi, p) = \sum_{i=1}^{r} \frac{1}{2} \int \int e^{-|\varphi(x)-\varphi(y)|^2} p(X_i(\varphi(y)))p(X_i(\varphi(x)))dxdy$$

Remark: When $r = d$ and $X_i = \frac{\partial}{\partial x_i}$ we get the diagonal Gaussian kernel.

Plan

Definition of a shape space

A "shape space" is a rather vague concept. Let us give a rigorous and general definition which unifies most cases (pretty much every case except images). Let $\mathcal{S}$ be a Banach Manifold, and s_0 smallest integer greater than $d/2$. Let $\ell \in \mathbb{N}^*$ and assume that $\mathcal{D}^{s_0+\ell}(M)$ has a continuous action $(q, \varphi) \mapsto \varphi \cdot q$ on $\mathcal{S}$. Denote $s = s_0 + \ell$.

Definition 3

$\mathcal{S}$ is a shape space of order $\ell \in \mathbb{N}^*$ if :

1. *The mapping $\varphi \mapsto \varphi \cdot q$ is smooth and Lipshitz for every q. Its differential at $\mathrm{Id}_{\mathbb{R}^d}$ is denoted $\xi_q : H^s(\mathbb{R}^d, \mathbb{R}^d) \to T_q\mathcal{S}$ and is called the infinitesimal action.*
2. *The mapping $\xi : \mathcal{S} \times H^{s+k}(\mathbb{R}^d, \mathbb{R}^d) \to T\mathcal{S}$ is of class $\mathcal{C}^k$.*

A state q has *compact support* if there exists $U \subset \mathbb{R}^d$ compact such that $\varphi \cdot q$ only depends on $\varphi_{|U}$.

Exemples

1. $\mathcal{D}^{s_0+\ell}(M)$ is a shape space of order ℓ for the composition on the left.
2. Let S be a compact manifold. Then $\mathrm{Emb}^\ell(S, \mathbb{R}^d)$ is a shape space of order ℓ for the action by left composition $(\varphi, q) \mapsto \varphi \circ q$.
3. For $\dim(S) = 0$, $S = \{s_1, \dots, s_n\}$ and $\mathrm{Emb}^\ell(S, \mathbb{R}^d) \simeq Lmk^n(\mathbb{R}^d)$ is a shape space of order 0, and $\varphi \cdot (x_1, \dots, x_n) = (\varphi(x_1), \dots, \varphi(x_n))$.
4. A product of shape spaces is a shape space.

Shape spaces of higher order

The tangent bundle of a shape space of order ℓ is a shape space of order $\ell + 1$. For example, for $\mathcal{S} = Lmk^n(\mathbb{R}^d)$, $T\mathcal{S} = Lmk^n(\mathbb{R}^d) \times (\mathbb{R}^d)^n$ and $\varphi \cdot (x_1, \dots, x_n, w_1, \dots, w_n)$ is given by

$$(\varphi(x_1), \dots, \varphi(x_n), d\varphi(x_1)w_1, \dots, d\varphi(x_n)w_n).$$

This example can be used to model muscles: w_i would be the direction of the muscle fiber at x_i.

Induced sub-Riemannian structure

Let $\mathcal{S}$ be a shape space of order ℓ, and V a Hilbert space of vector fields with continuous inclusion in $H^{s+k}(\mathbb{R}^d, \mathbb{R}^d)$. Then $\xi : (q, X) \mapsto \xi_q X$ and $\langle \cdot, \cdot \rangle$ define a sub-Riemannian structure on $\mathcal{S}$ of class $\mathcal{C}^k$.

A curve $t \mapsto q(t)$ is horizontal if there exists $X \in L^2(0, 1; V)$,

$$\dot{q}(t) = \xi_{q(t)} X(t).$$

In other words, $q(t) = \varphi^X(t) \cdot q(0)$ where $t \mapsto \varphi^X(t)$ is the flow of X. The energy of q is $\frac{1}{2} \int_0^1 \langle X(t), X(t) \rangle \, dt$. We then define the sub-Riemannian length, the sub-Riemannian distance d and geodesics as usual.

Sub-Riemannian distance

Proposition 2

The sub-Riemannian distance is a true distance with values in $[0, +\infty]$.
Let $q_0 \in \mathcal{S}$ *have compact support, and* $\mathcal{O}_{q_0} = \{q \in \mathcal{S} \mid d(q_0, q) < +\infty\}$. *Then* $(\mathcal{O}_{q_0}, d)$ *is a complete metric space, and any two points can be connected by a geodesic.*

Hamiltonian equations

The Hamiltonian $H : T^*\mathcal{S} \to \mathbb{R}$ is

$$H(q,p) = \max_{X \in V} \; p\xi_q X - \frac{1}{2}\langle X, X\rangle = \frac{1}{2} p\xi_q K_V \xi_q^* p.$$

Denoting $K_q = \xi_q K_V \xi_q^* : T_q^*\mathcal{S} \to T_q\mathcal{S}$, we get $H(q,p) = \frac{1}{2} pK_q p$.

Proposition 3

Let $t \mapsto (q(t), p(t))$ satisfy the Hamiltonian equations

$$\dot{q} = \partial_p H(q(t), p(t)), \quad \dot{p} = -\partial_q H(q(t), p(t)).$$

Then $q(\cdot)$ is a geodesic on small enough intervals.

The converse is *not* true.

LDDMM

Proposition 4

Let $t \mapsto X(t) \in V$ minimize

$$J(X) = \int_0^1 \langle X(t), X(t) \rangle \, dt + g(q(1)),$$

where $q(t) = \varphi^X(t) \cdot q_0$, q_0 fixed, and $g : \mathcal{S} \to \mathbb{R}$ of class $\mathcal{C}^1$. Then there exists $t \mapsto p(t) \in T^*_{q(t)}\mathcal{S}$ such that $(q(\cdot), p(cdot))$ satisfy the Hamiltonian equations.