Constrained shape spaces of infinite dimension

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Goal: Adding constraints to shapes in order to better fit the modeled object. (Ex: Shapes with constant volume, several independent shapes that can't overlap...)
Finding geodesic equations and minimization algorithms.

Method: Optimal control in finite and infinite dimensions.

Plan

1. Framework
    - Deformations and cost function
    - Constraints
2. Existence and properties of minimizers

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Deformations

Let S be a compact Riemannian manifold and $M = \mathcal{C}^0(S, \mathbb{R}^d)$ be our shape space.

Exemples:

- When $S = S^1$ the unit circle, M is the set of all continuous closed curves.
- When $dim(S) = 0$, $S = (s_1, \ldots, s_n)$ is a union of n points: $M = (\mathbb{R}^d)^n$ is the landmark space.

Action of the group of diffeomorphisms

Left action: φ a diffeomorphism of class $\mathcal{C}^1$ acts on $q \in M$ by $\varphi \cdot q = \varphi \circ q$.

On landmarks: $\varphi \cdot (x_1, \dots, x_n) = (\varphi(x_1), \dots, \varphi(x_n))$.

Infinitesimal action: v vector field of class $\mathcal{C}^1$ acts on M by $v \cdot q = v \circ q \in T_q M$.

On landmarks: $v \cdot (x_1, \dots, x_n) = (v(x_1), \dots, v(x_n))$.

Momentum map: this map $T^*M \to \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)^*$ is written $(q, p) \to \mu(q, p)$, with

$$\mu(q, p)(v) = p(v \cdot q).$$

On landmarks: $\mu(x_1, \dots, x_n, p_1, \dots, p_n) = p_1 \otimes \delta_{x_1} + \cdots + p_n \otimes \delta_{x_n}$.

Cost function

Let $(V, \langle\cdot,\cdot\rangle)$ be an RKHS of bounded vector fields, with kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to M_n(\mathbb{R})$. Also denote by K_V the isometry $V^* \to V$.

Cost function: for a fixed starting point $q_0 \in M$ and a data attachment function $g : M \to \mathbb{R}$, we will want to minimize the cost function $J : L^2([0,1], V) \longrightarrow \mathbb{R}$ given by

$$J(v) = \frac{1}{2}\int_0^1 \|v(t)\|^2 dt + g(q(1)),$$

with $q(1)$ given by the control system

$$q(0) = q_0 \quad \text{and} \quad \dot{q}(t) = v(t) \circ q(t).$$

Remark

In finite dimensions, if v is of minimal norm among all vector fields with the same infinitesimal action on the state q, then for some p,

$$v = K_V \mu(q,p) = K(\cdot, x_1)p_1 + \cdots + K(\cdot, x_n)p_n.$$

Not true in infinite dimensions. (if range of μ is not closed)

Semi-linear constraints

Addition of constraints: Let Y be a Banach space. Define constraints $C : M \times V \to Y$ linear in the second variable, i.e. of the form $C(q, v) = C_q v$.
We will want to restrict ourselves to constrained trajectories: $C_{q(t)} v(t) = 0$.

Particular cases:

1. "Kinetic" constraints, i.e. constraints on the speed: $C_q \dot{q} = C_q (v \cdot q)$.
2. State constraints $C(q) = 0$ are equivalent to the constraints $dC_q (v \cdot q) = 0$.

Multishapes

Consider several shapes $S_1, \dots, S_k$, boundaries of open sets $U_1, \dots, U_k$ of $\mathbb{R}^d$, and $S = S_1 \sqcup \cdots \sqcup S_k$, boundary of $U = (U_1 \cup \cdots \cup U_k)^c$.
Each U_i: distinct shape, deformed by a distinct diffeomorphism φ_i.
U: "background", deformed by another difféomorphism φ.
We want all boundaries to keep glued together along the deformation:

$$\varphi_{i|S_i} = \varphi_{|S_i}, \quad i = 1 \dots, k.$$

Application: studying simultaneously different parts of the brain.

Multiformes

If we want the constraints to be independent of the parametrisation of the boundaries, we need to allow sliding constraints, reducing the constraints to

$$\varphi_i(S_i) = \varphi(S_i), \quad i = 1 \ldots, k.$$

Remark: This is a pure state constraint, but its discretization to landmarks is not.

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1 Framework
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2 Existence and properties of minimizers

Statement of the problem

Optimal control problem: We want to minimize

$$J(v) = \frac{1}{2}\int_0^1 \|v(t)\|^2 dt + g(q(1)),$$

with $q(1)$ given by the control system

$$q(0) = q_0 \quad \text{and} \quad \dot{q}(t) = v(t)\cdot q(t),$$

among all $v \in L^2([0,1], V)$ such that

$$C_{q(t)}v(t) = 0 \quad a.e.\ t \in [0,1].$$

Existence

Theorem 1

Assume

1. *The norm on V satisfies* $\|v\|_{1,\infty} \leq c\|v\|$,
2. *C est continuous,*
3. *g est continuous and admits a lower bound.*

Then J admits a constrained minimizer $v^* \in L^2([0,1], V)$.

Hamiltonien

Hamiltonian: It is defined on $T^*M \times V \times Y^*$ by

$$H(q,p,v,\lambda) = p(v \circ q) - \frac{1}{2}\|v\|^2 - \lambda(C_q v),$$

with $q \in M$, $p \in M^*$ (Lagrange multiplier of $\dot{q} = v \circ q$), $v \in V$ and $\lambda \in Y^*$ ((Lagrange multiplier of $C_q v = 0$).

Maximum principle

Theorem 2

Assume that

1. $\|v\|_{1,\infty} \leq c\|v\|$,
2. *C and g are of class $\mathcal{C}^1$,*
3. C_q **is onto for every** q. *(restrictive)*

Then, for every constrained minimizer, there exists $p \in H^1([0,1], M^*)$ and $\lambda \in L^2([0,1], Y^*)$ such that $p(1) + dg_{q(1)} = 0$ and a.e. $t \in [0,1]$,

$$\begin{cases} 0 &= \partial_v H(q, p, v, \lambda) \\ \dot{q} &= \partial_p H(q, p, v, \lambda) \\ \dot{p} &= -\partial_q H(q, p, v, \lambda) \end{cases}$$

Moreover, $t \mapsto \|v(t)\|$ is constant.

Reduction

First equation gives

$$\partial_v H(q,p,v,\lambda) = 0 \quad \Rightarrow \quad v = v(q,p) = K_V(\mu(q,p) - C_q^*\lambda).$$

$C_q v = 0$ implies

$$\lambda = \lambda(q,p) = (C_q K_V C_q^*)^{-1} C_q K_V \mu(q,p).$$

Remark: In particular, when the constraints are kinetic, the first equation becomes

$$v = K_V \mu(q, p - C_q^*\lambda) = K_V \mu(q,u).$$

Geodesic equations

We get the constrained geodesic equations

$$\begin{cases} \dot{q} & = \partial_p h(q,p), \\ \dot{p} & = -\, \partial_q h(q,p), \end{cases}$$

with $h(q,p) = H(q,p,v(q,p),\lambda(q,p))$ is the reduced hamiltonian and $t \mapsto \|v(q(t),p(t))\|$ is constant.

Minimization algorithm: we just minimize

$$J(v(q,p)) = \frac{1}{2}\|v(q(0),p(0))\|^2 + g(q(1))$$

with respect to the initial momentum $p(0)$.

Thank you for your attention!

Theorem 3

Let $C^n : M \times V \to Y^n$ be a sequence of continuous constraints such that $\ker C_q^n$ *is decreasing and* $\bigcap_n \ker C_q^n = \ker C_q$.
Let $g^n : M \to \mathbb{R}$ be a sequence of continuous data attachment maps that converges uniformly to g on every compact subset of M, and let J^n the associated costs.
Let v^n be minimizers of J^n along the constraints defined by C^n.
Then, up to the extraction of a subsequence, v^n converges weakly toward a minimizer v^* of the original problem. Moreover, if there is only one such minimizer, there is no need to extract a subsequence.