Large Deformations and Triangulation for Image Matching Problems

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Objective and Context:

Given 2 images, $I_0$ a template and $I_1$ a target, we seek a function $\phi$ with smoothness properties such as: $\phi(I_0) \simeq I_1$

Our framework is the Deformable Template model (Grenander) in particular the Large Deformation Theory (Miller-Touvé-Younes)
Mathematical background and Notations:

Large Deformations

* Let $\Omega \in \mathbb{R}^d$ be an open set. The framework defines a class of deformations $\phi : \Omega \to \Omega$, objects (eg: images $I : \Omega \to \mathbb{R}$, landmarks $(x_i)_{1 \leq i \leq N, ...}$) and a specific groupe action (eg: $\phi.I = I \circ \phi^{-1}$, $\phi.( (x_i)_{1 \leq i \leq N}) = (\phi.(x_i))_{1 \leq i \leq N}$)

Deformations $\phi$ are built by integrating time-dependent vector field $v_t : \Omega \to \mathbb{R}^l$ (in our cases $l = d = 2, 3$) : $\phi = \phi_1^v$

$$\left\{ \begin{array}{l}
\frac{d\phi_t^v}{dt} = v_t \circ \phi_t^v \\
\phi_0 = Id
\end{array} \right.$$  

* $v_t \in V$ Hilbert space of regular vector field, in particular, $V$ is a functional space, continuously embedded in $C^0(\Omega)$, defined by an operateur $L$ and its reproducing kernel $K_V = L^{-1}$ such as :

$$\forall (v, w) \in V^2, \langle v, w \rangle_V = \langle K_V v, w \rangle_{L^2(\Omega)}$$
\* \{\phi^v_1, v \in L^2([0, 1], V)\} is a \textbf{subgroup of diffeomorphisms} on \Omega, equipped with a right-invariant metric: \(d(\phi, \psi) = d(Id, \psi \circ \phi^{-1})\).

\* The \textbf{distance} between two objects \(O_0\) and \(O_1\) is computed via the group action:

\[
d(O_0, O_1) = \inf_{v_t \in V, \phi^v_1(O_0) = O_1} d(Id, \phi^v_1) = \inf_{v_t \in V, \phi^v_1(O_0) = O_1} \{ \int_0^1 \|v_t\|^2_V dt \}
\]
Method to find the velocity vector field

We seek $v_t$ by minimizing an energy that takes into account two terms: the path length (given by the previous formula) and a measurement of the difference between our data:

$$v_t = \arg \min \left\{ \frac{1}{2} \int_0^1 \|v_t\|^2 dt + \lambda g(O_0, O_1, \phi^\nu) \right\}$$

2 different approaches

*Image matching* (Beg) Given 2 images $I_0$ and $I_1$, $v_t$ is determined everywhere on the domain for every time $t$ by minimizing:

$$E = \frac{1}{2} \int_0^1 \|v_t\|^2 dt + \lambda \int_\Omega |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy$$
Landmark Matching (Joshi - Miller) Given several template landmarks \( (x_i)_{1 \leq i \leq N} \) and target landmarks \( (y_i)_{1 \leq i \leq N} \), \( v_t \) is determined by minimizing the following energy:

\[
E = \frac{1}{2} \int_0^1 \|v_t\|^2_V dt + \lambda \sum_{i=1}^N \|\phi^v(x_i) - y_i\|^2_{\mathbb{R}^k}.
\]

that in fact depends only on a new variable:
\( (p_i(t) = L(v_i(t)))_{1 \leq i \leq N} \), momentum of the deformation at time \( t \).

To reconstruct \( v_t \) on the whole domain, we use the interpolation formula:

\[
\forall x \in \Omega, \ v_t(x) = \sum_{i=1}^N K(x_i(t), x)p_i(t)
\]

Our approach is a combination between these 2 points of view
Our model:

Data 2 images, a template $I_0$, a target $I_1$ and landmarks chosen on the template $(x_i)_{1 \leq i \leq N}$.

Wanted $\phi_v$ that matches the template on the target but only dependent on the landmark set.

Idea: A triangulation of the template, an affine transformation on each triangle and continuous on the whole domain.

This yields that the deformation is only determined by the vertices of the triangulation (our landmarks).

Resulting Energy:

$$E = \frac{1}{2} \int_0^1 \|v_t\|_V^2 dt + \lambda \sum_{i=1}^r \int_{\phi(T_i)} |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy$$
Triangulation examples

We use the Delaunay’s triangulation of a point set:
**Problem Reformulation:**

**Conservation of Momentum Property**

Using the “conservation of Momentum” property of the geodesics (Miller - Trouvé - Younes) the momentum at time $t$ is determined by the momentum configuration at time $t = 0$: **Momentum Evolution Equation**: $p_t(x) = L v_t(x) = L v_0((d\phi_t)^{-1}x \circ \phi_t)$

and the **Euler’s equation** of geodesics: $p_1(p_0) + \lambda \nabla_{x} g = 0$

**Consequence**: equation for geodesic evolution depends only on the template and the momentum at time 0

So $p_0$ is an appropriate variable of the problem.
Hamiltonian framework

Evolution equations describing the transport of the template along the geodesics (cf: M.I. Miller A. Trouvé L. Younes): let $q$ be the point and $p$ the momentum: **Hamilton’s Equations**

\[
\begin{align*}
\frac{dq_i(t)}{dt} &= \sum_{j=1}^{N} K(q_j(t), q_i(t)) p_j(t) = K(q_i(t)) p(t) \\
\frac{dp_i(t)}{dt} &= -(d_{q_i(t)} v_t)^* p_i(t)
\end{align*}
\]
With these equations, we search the best initial conditions that give rise to the minimizing trajectory.

The path length term can be rewritten:

\[ d = \frac{1}{2} \int_0^1 \|v_t\|^2_V dt = \frac{1}{2} \int_0^1 \frac{dq^*}{dt} K(q(t))^{-1} \frac{dq}{dt} dt \]

therefore \( \frac{dq}{dt} = K(q(t))p(t) \) then: \( d = \frac{1}{2} \int_0^1 p(t)^* K(q(t))p(t) dt \) where \( H(q(t), p(t)) = \frac{1}{2} p(t)^* K(q(t))p(t) \) is known as the Hamiltonian.

**Property of the Hamiltonian**: \( H(q(t), p(t)) \) is a constant function of time.
So we finally have the **Hamiltonian system** :

\[
\begin{align*}
\frac{dq(t)}{dt} &= K_\sigma p \\
\frac{dp(t)}{dt} &= \frac{1}{2}\langle K_\sigma p, p \rangle
\end{align*}
\]

the **Euler’s equation** : 
\[p_1(p_0) + \lambda \nabla_x g = 0\]

And the energy to minimize is :

\[E = \frac{1}{2} p(0)^* K(q(0)) p(0) + \lambda \sum_{i=1}^{r} \int_{\phi(T_i)} |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy\]

with respect to \(p_0\).
Algorithms:

Gradient descent

The gradient descent is computed in the initial momentum space (cf: M. Vaillant M.I. Miller L. Younes A. Trouvé).

Energy to minimize:

\[ E = \frac{1}{2} p(0) \ast K(q(0)) p(0) + \lambda \sum_{i=1}^{r} \int_{\phi(T_i)} |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy \]

Algorithm:

Let \( g(x) \) be the data attachment term, and \( q_i^1 = \phi_1(x_i) \).

\[
p_0^{k+1} = p_0^k - \alpha \nabla_{p_0} E = p_0^k - \alpha (K(q_0)p_0 + \lambda \frac{dq_0}{dq} \frac{dq^1}{dp_0})
\]
Newton’s method

We solve the Euler’s Equation:

\[ G(p_0) = p_1(p_0) + \lambda \nabla_{q^1} g(q^1(p_0)) = 0 \]

Algorithm:

\[
p^{k+1}_0 = p^k_0 - (d_{p_0} G)^{-1} G(p^k_0)
= p^k_0 - (\frac{dp_1}{dp_0} + \lambda \frac{d^2 g}{dq^1 dp_0} \frac{dq^1}{dp_0})^{-1}(p_1(p_0) + \lambda \nabla_x g(q_1(p_0)))
\]
Gradient Computation

The gradient and the second derivative of $g$ are needed. $g(q^1) = \sum_{i=1}^{r} \int_{\phi(T_i)} |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy = \sum_{i=1}^{r} \int_{T_i} |I_0(x) - I_1(\phi(x))|^2 |d_x \phi| dx$ As $\phi$ is sought affine by part we can use the barycentric coordinates:

Let $S$ be the ideal simplex $((0,0), (1,0), (0,1))$ and $M_i, M'_i$ be the 2 linear applications:

$$
\begin{align*}
    S & \mapsto M'_i(S) = \phi_v(T_i) \\
    S & \mapsto M_i(S) = T_i
\end{align*}
$$

And $q_{\varepsilon,i}(\alpha, \beta) = x^1_{\varepsilon,i} + \alpha(x^2_{\varepsilon,i} - x^1_{\varepsilon,i}) + \beta(x^3_{\varepsilon,i} - x^1_{\varepsilon,i})$, for $\varepsilon = 0, 1$, and $1 \leq i \leq N$.

Then, $x = M_i(\alpha, \beta)$, $\phi(x) = M'_i(\alpha, \beta) = M'_i(M_i^{-1}(x))$ and $|d_x \phi| = |M'_i M_i^{-1}|$. 
Thus:

\[ g(\mathbf{z}) = \sum_{i=1}^{r} \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} |I_1(q_1, i(\alpha, \beta)) - I_0(q_0, i(\alpha, \beta))|^2 |M'_i| d\alpha d\beta \]

where \( \mathbf{z} = (\phi_v(x_1), ..., \phi_v(x_N))^T \). Let \(|A_i| = |M'_i|\).

And its gradient:

\[
\frac{\partial g}{\partial \mathbf{z}} = \sum_{i=1}^{r} \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} 2(I_1(q_1, i(\alpha, \beta)) - I_0(q_0, i(\alpha, \beta))) |A_i(z_i)| \\
\quad \times (\partial_{z_i} q_1, i(\alpha, \beta))^* \nabla I_1(q_1, i(\alpha, \beta)) d\alpha d\beta \\
+ \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} |I_1(q_1, i(\alpha, \beta)) - I_0(q_0, i(\alpha, \beta))|^2 \partial_{z_i} (|A_i(z_i)|) d\alpha d\beta
\]
Second derivative

We also get for a fixed triangle $T_i$:

$$\frac{\partial^2 q_i}{\partial z^2} = $$

$$\int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} 2(\delta z_2)^* (\delta z q_1,i) \nabla q_1,i I_1(\nabla q_1,i I_1)^* (\delta z q_1,i) \delta z_1 |A_i(z_i)| d\alpha d\beta$$

$$+ \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} 2(\delta z_2)^* (\delta z q_1,i)^* (\partial_{q_1,i,q_1,i}^2 I_1)(\delta z q_1,i) \delta z_1 |A_i(z_i)|$$

$$(I_1(q_1,i(\alpha,\beta)) - I_0(q_0,i(\alpha,\beta))) d\alpha d\beta$$

$$+ \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} 2(I_1(q_1,i(\alpha,\beta)) - I_0(q_0,i(\alpha,\beta)))(\delta z_2)^* (\delta z q_1,i)^*$$

$$\nabla q_1,i I_1(\nabla z |A_i(z_i)|)^* \delta z_1 d\alpha d\beta$$

$$+ \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} (I_1(q_1,i(\alpha,\beta)) - I_0(q_0,i(\alpha,\beta))^2(\delta z_2)^* \partial_{z,z}^2 |A_i(z_i)| \delta z_1 d\alpha d\beta$$
Experiments

Template

Target

phi(10)

phi
Common mesh for all images
Limitations of each algorithm

Gradient descent
* The convergence speed

Newton’s method
* The initialization point
* The matrix conditionnement.

Solution Projection on the main singular directions of the matrix before inversion.

Both
* The triangle consistency to keep an homeomorphic deformation
Remark: Landmark matching

Using the same point of view, we can do landmark matching as well. The energy is given by:

\[ E = \frac{1}{2} \int_0^1 \|v_t\|_V^2 \, dt + \lambda \sum_{i=1}^{N} \|q_i^1 - y_i\|_{\mathbb{R}^k}^2. \]

the data attachment term derivatives equal \( \frac{dq}{dq^1} = 2(q_i^1 - y_i) \) and
\( \frac{d^2 q}{(dq^1)^2} = 2Id_{N \times d} \), where \( d \) is the dimension.
Conclusion

* The triangulation is a way to reduce the system dimension, focusing on landmark evolutions.
* Can be generalized for 3-D images.
* Newton’s method has the advantage of speed of convergence.
Automatic landmark detection

Let $w$ be a window. The energy to minimize is:

$$E = \frac{1}{2} \int_0^1 \|v_t\|^2_V dt + \lambda \sum_{i=1}^N \int_{\Omega} |I_0 \circ \phi_1^{-1}(y) - I_1(y)|^2 w(x_i - \phi_1^{-1}(y))dy$$