Recursive Identification of Switched ARX Hybrid Models: Exponential Convergence and Persistence of Excitation

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***Abstract*— We propose a recursive identification algorithm for a class of discrete-time linear hybrid systems known as Switched ARX models. The key to our approach is to view the identification of multiple ARX models as the identification of a single, though more complex, lifted dynamical model in a higher dimensional space. Since the dynamics of this lifted model do not depend on the value of the discrete state or the switching mechanism, we propose to use a standard recursive identifier in the lifted space. We derive persistence of excitation conditions on the input/output data that guarantee the exponential convergence of the recursive identifier. Such conditions are a natural generalization of the well known result for ARX models. We then use the estimates of the lifted model parameters to build a homogeneous polynomial whose derivatives at a regressor give an estimate of the parameters of the ARX model generating that regressor. Although our algorithm is designed for the case of perfect input/output data, our experiments also show its performance with noisy data.**

I. Introduction

Given input/output data generated by a hybrid linear dynamical model, *i.e.* a system that switches among an unknown number of linear dynamical models, we are interested in simultaneously identifying the number of models, the model parameters, the mode sequence (*i.e.* the model generating the data at each time) and the switching mechanism generating the transitions among the linear models.

When the model parameters and the switching mechanism are *known*, the above problem reduces to the design of observers for the hybrid state [1], [5], [11], [14], together with the study of observability conditions under which hybrid observers operate correctly [4], [6], [9], [16], [17].

When the model parameters and the switching mechanism are *unknown*, there is an apparent coupling between the tasks of identifying the model parameters and estimating the discrete state, which makes the identification problem significantly more challenging. Due to this coupling, most of the existing hybrid identification algorithms alternate between parameter identification and state estimation. For instance, [12] assumes that the number of models is known, and proposes an identification algorithm that combines clustering, regression and classification techniques; [7] uses a greedy approach for initialization and then iterates between assigning data points to models and computing the model parameters; [8] uses mixed-integer quadratic programming.

Our recent work [19], [15] demonstrated that for the class of switched ARX models one can derive an equation on the input/output data called the *hybrid decoupling polynomial* that does not depend on the mode sequence or the switching mechanism. We also showed that one can identify the number of modes from a rank constraint on the input/output data and the model parameters from the derivatives of the hybrid decoupling polynomial at a data point.

Unfortunately, all the aforementioned hybrid identification algorithms operate in a batch mode, *i.e.* the model parameters and the hybrid state are identified after all the input/output data has been collected. This is a significant limitation, because the computational complexity of batch algorithms depends on the number of data points, hence they may not be suitable for real time applications. To the best of our knowledge, there is no prior work addressing the *recursive* identification of hybrid dynamical models in the case in which the model parameters, the hybrid state and the switching mechanism are unknown.

In this paper we propose a recursive algorithm for identifying the model parameters of switched ARX models whose mode sequence and switching mechanism are unknown. Rather than defining a recursive identifier of the parameters of each ARX model, we propose to identify the *hybrid model parameters* of a lifted model in a higher dimensional space. A similar idea was used in [10] for the identification of ARX models with structured parameters. Since the dynamics of the lifted model are linear on the hybrid model parameters, one can define convergence in the lifted space and derive persistence of excitation conditions on the input/output data that guarantee the exponential convergence of the hybrid model parameter estimates. Our conditions are a natural generalization of those for a single ARX model. However, they also impose some restrictions on the mode sequence. For instance, they require each mode to be visited a minimum number of times. Given an exponentially convergent estimate of the hybrid model parameters, we build a homogeneous polynomial whose derivatives give an exponentially convergent estimate of the parameters of the ARX models. Although our identification algorithm is derived for the case of perfect input/output data, our experiments also show its performance with noisy data.

II. PROBLEM STATEMENT

In this paper we consider a class of discrete-time linear hybrid systems known as Switched Auto Regressive eXogenous (SARX) models. The temporal evolution of a SARX model is described as

$$y_t = \sum_{j=1}^{n_a} a_j(\lambda_t) y_{t-j} + \sum_{j=1}^{n_c} c_j(\lambda_t) u_{t-j}, \tag{1}$$

where $u_t \in \mathbb{R}$ is the *input*, $y_t \in \mathbb{R}$ is the output, $\lambda_t \in \{1, 2, \ldots, n\}$ is the *discrete state*, n_a and n_c are the orders of each ARX model, and $\{a_\ell(i)\}_{\ell=1}^{n_a}$ and $\{c_\ell(i)\}_{\ell=1}^{n_c}$, for $i = 1, \ldots, n$, are their model parameters. We assume that $n_c > 0$ and that all the ARX models are of known and equal orders, *i.e.* for all $i = 1, \ldots, n$ either $a_{n_a}(i) \neq 0$ or $c_{n_c}(i) \neq 0$. However, it should be possible to extend our approach to ARX models of unknown and different orders by combining our results with the batch algorithm of [15].

The discrete state λ_t, also called the *mode* of the system, can evolve due to a variety of mechanisms. In the least restrictive case, $\{\lambda_t\}$ is a deterministic but unknown sequence that can take a finite number of possible values, which we can assume to coincide with a collection of integers: $\lambda : \mathbb{Z} \to \{1, \ldots, n\}$. One can further restrict the set of mode sequences by assuming that λ_t is a realization of an irreducible Markov chain, governed by transition probabilities $\pi(i,j) \doteq P(\lambda_{t+1} = j | \lambda_t = i)$. In this case, the model (1) is often called a "Jump-Markov Linear System" (JMLS). Finally, one can assume that λ_t is a piecewise constant function of the evolution of the model (1), $\lambda_t(y_t, y_{t-1}, \ldots, y_{t-n_a})$. In this case, the model (1) is often called a "PieceWise ARX" (PWARX) model.

In this paper we will consider the first scenario, so that our results also apply to other switching mechanisms if that information becomes available. The following problem summarizes the goals of this paper:

Problem 1: Let $\{u_t, y_t\}_{t=0}^T$ be input/output data generated by the SARX model (1), with known number of discrete states n and orders n_a and n_c. Identify the model parameters $\{a_\ell(i)\}_{\ell=1,\ldots,n_a}^{i=1,\ldots,n}$ and $\{c_\ell(i)\}_{\ell=1,\ldots,n_c}^{i=1,\ldots,n}$, and estimate the discrete state $\{\lambda_t\}_{t=\max\{n_a,n_c\}}^T$.

The next sections present a solution to Problem 1 under the following assumption of minimality of the SARX model.

Definition 1: A SARX model is said to be *minimal* if

1) For all $i = 1, \ldots, n$, the ith ARX model is minimal. That is, the numerator and denominator of the transfer function $H_i(z)$ of the ith ARX model are coprime polynomials, hence there is no zero/pole cancelation.
2) For all $i \neq j = 1, \ldots, n$ the transfer functions $H_i(z)$ and $H_j(z)$ are such that $H_i(z) \neq H_j(z)$ for all $z \in \mathbb{C}$.

III. THE HYBRID DECOUPLING POLYNOMIAL AND THE HYBRID MODEL PARAMETERS

In this section we review the derivation of the hybrid decoupling polynomial and its associated vector of hybrid model parameters. We refer the reader to [19] for further details.

A. The hybrid decoupling polynomial

Notice from equation (1) that if we let $K \doteq n_a + n_c + 1$,

$$\boldsymbol{x}_t = [u_{t-n_c}, \ldots, u_{t-1}, y_{t-n_a}, \ldots, y_{t-1}, -y_t]^T \in \mathbb{R}^K \text{ and}$$
$$\boldsymbol{b}_i = [c_{n_c}(i), \ldots, c_1(i), a_{n_a}(i), \ldots a_1(i), 1]^T \in \mathbb{R}^K,$$

for $i = 1, \ldots, n$, then we have that for all $t \geq \max\{n_a, n_c\}$ there exists a discrete state $\lambda_t = i \in \{1, \ldots, n\}$ such that

$$\boldsymbol{b}_i^T \boldsymbol{x}_t = 0. \tag{2}$$

Therefore, the following *hybrid decoupling polynomial* must be satisfied by the model parameters and the input/output data for any possible value of the discrete state and for any possible switching mechanism generating the evolution of the discrete state (SARX, JMLS, or PWARX)

$$\prod_{i=1}^n (\boldsymbol{b}_i^T \boldsymbol{x}_t) = 0. \tag{3}$$

B. The hybrid model parameters

The hybrid decoupling polynomial eliminates the discrete state by taking the product of the equations defining each one of the ARX models. While taking the product is not the only way of algebraically eliminating the discrete state, this leads to an algebraic equation with a very nice algebraic structure. Indeed, the hybrid decoupling polynomial is simply is a homogeneous polynomial of degree n in K variables

$$p_n(\boldsymbol{z}) \doteq \prod_{i=1}^n (\boldsymbol{b}_i^T \boldsymbol{z}) = 0, \tag{4}$$

which can be written as

$$p_n(\boldsymbol{z}) \doteq \sum h_{n_1,\ldots,n_K} z_1^{n_1} \cdots z_K^{n_K} = \boldsymbol{h}^T \nu_n(\boldsymbol{z}) = 0, \tag{5}$$

where $h_I \in \mathbb{R}$ represents the coefficient of the monomial $\boldsymbol{z}^I = z_1^{n_1} z_2^{n_2} \cdots z_K^{n_K}$ with $0 \leq n_j \leq n$ for $j = 1, \ldots, K$, and $n_1 + n_2 + \cdots + n_K = n$; $\nu_n : \mathbb{R}^K \to \mathbb{R}^{M_n(K)}$ is the *Veronese map* of degree n which is defined as [13]:

$$\nu_n : [z_1, \ldots, z_K]^T \mapsto [\ldots, \boldsymbol{z}^I, \ldots]^T, \tag{6}$$

with I chosen in the degree-lexicographic order; and

$$M_n(K) = \binom{n+K-1}{K-1} = \binom{n+K-1}{n} \tag{7}$$

is the total number of independent monomials.

One can show (see [13]) that the vector $\boldsymbol{h} \in \mathbb{R}^{M_n(K)}$ is simply a vector representation of the symmetric tensor product of the ARX model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$, *i.e.*

$$\sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{b}_{\sigma(1)} \otimes \boldsymbol{b}_{\sigma(2)} \otimes \cdots \otimes \boldsymbol{b}_{\sigma(n)}, \tag{8}$$

where $\mathfrak{S}_n$ is the permutation group of n elements. Therefore, since the vector $\boldsymbol{h} \in \mathbb{R}^{M_n(K)}$ encodes the parameters of all the ARX models, we will refer to it as the *hybrid model parameters* from now on. Notice that the last entry of $\boldsymbol{h}$ is always one, because the last entry of each $\boldsymbol{b}_i$ is also one. Therefore, $\boldsymbol{h}$ is uniquely defined and there is a one-to-one correspondence between $\boldsymbol{h}$ and the ARX model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$ modulo a permutation of the latter ones.

IV. Recursive Identification of the Hybrid Model Parameters

In this section we propose a recursive algorithm for identifying the hybrid model parameters and derive a persistence of excitation condition that guarantees the exponential convergence of the estimates of the hybrid model parameters.

Recall from [2] that in the case of a single minimal ARX model, *i.e.* if $n = 1$, the recursive *equation error identifier*

$$\hat{\boldsymbol{b}}_{t+1} = \hat{\boldsymbol{b}}_t - \mu \begin{bmatrix} \frac{\Pi_1 \boldsymbol{x}_t (\hat{y}_t - y_t)}{1+\mu\left(\sum_{j=1}^{n_a} y_{t-j}^2 + \sum_{j=1}^{n_c} u_{t-j}^2\right)} \\ 0 \end{bmatrix} \tag{9}$$

$$\hat{y}_t = \hat{\boldsymbol{b}}_t^T \Pi_1^T \Pi_1 \boldsymbol{x}_t = \sum_{j=1}^{n_a} \hat{a}_{jt} y_{t-j} + \sum_{j=1}^{n_c} \hat{c}_{jt} u_{t-j}, \tag{10}$$

where $\Pi_1 = [I_{K-1} \ 0] \in \mathbb{R}^{(K-1)\times K}$ and $\mu > 0$, produces an exponentially convergent estimate of the model parameters $\boldsymbol{b} \in \mathbb{R}^K$ if the input/output data is *persistently exciting*, *i.e.* if there is an $S \in \mathbb{N}$ and $\rho_1, \rho_2 > 0$ such that for all j

$$\rho_1 I_{K-1} \prec \sum_{t=j}^{j+S} \Pi_1 \boldsymbol{x}_t \boldsymbol{x}_t^T \Pi_1^T \prec \rho_2 I_{K-1}, \tag{11}$$

where $A \prec B$ if $(B - A)$ is positive definite. It is shown in [3] that the above condition for exponential convergence is satisfied if the input sequence is *persistently exciting*, *i.e.* if there is an $S \in \mathbb{N}$ and $\rho_3, \rho_4 > 0$ such that for all j

$$\rho_3 I_{K-1} \prec \sum_{t=j}^{j+S-n_a+1} \boldsymbol{u}_t \boldsymbol{u}_t^T \prec \rho_4 I_{K-1}, \tag{12}$$

where $\boldsymbol{u}_t = [u_{t-n_c}, \ldots, u_{t+n_a-1}]^T \in \mathbb{R}^{K-1}$.

The question is now how to generalize the recursive identifier in (9)-(10) and its convergence properties to the case of SARX models such as (1). In principle, we could try using the same recursive identifier (9)-(10). However, characterizing its exponential convergence is not as straightforward, because its dynamics are now hybrid and each discrete state may only be visited a finite amount of time.

Thanks to the hybrid decoupling polynomial, we can now propose a new recursive identification algorithm that operates on the hybrid model parameters $\boldsymbol{h}$ rather than on the ARX model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$. The advantage of doing so is that the hybrid model parameters do not depend on the value of the discrete state or the switching mechanism. Therefore, we can directly propose a recursive identifier $\hat{\boldsymbol{h}}_t$ for the hybrid model parameters $\boldsymbol{h}$. Given an estimate for $\boldsymbol{h}$, the ARX model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$ can be easily identified, as we will show in Section V.

To this end, let $\Pi_n = [I_{M_n(K)-1} \ 0] \in \mathbb{R}^{(M_n(K)-1)\times M_n(K)}$ and consider the recursive identification scheme

$$\hat{\boldsymbol{h}}_{t+1} = \hat{\boldsymbol{h}}_t - \mu \begin{bmatrix} \frac{\Pi_n \nu_n(\boldsymbol{x}_t)(\hat{\boldsymbol{h}}_t^T \nu_n(\boldsymbol{x}_t))}{1+\mu\|\Pi_n \nu_n(\boldsymbol{x}_t)\|^2} \\ 0 \end{bmatrix}. \tag{13}$$

Notice that (13) reduces to (9) if $n = 1$. The following theorem gives a sufficient condition for the exponential convergence of the *hybrid equation error identifier* (13) in terms of the regressors $\{\boldsymbol{x}_t\}$ generated by the SARX model.

Theorem 1 (Persistence of excitation for SARX models): Consider a minimal SARX system of the form (1) and assume that the recursive identification scheme (13) is used. If there exist $\rho_1, \rho_2 > 0$ and an integer S such that for all $j \geq \max\{n_a, n_c\}$

$$\rho_1 I_{M_n(K)-1} \prec \sum_{t=j}^{j+S} \Pi_n \nu_n(\boldsymbol{x}_t) \nu_n^T(\boldsymbol{x}_t) \Pi_n^T \prec \rho_2 I_{M_n(K)-1}, \tag{14}$$

then $\boldsymbol{h} - \widehat{\boldsymbol{h}}_t$ converges exponentially to zero.

Proof: The proof of this result for ARX models, can be found, *e.g.* in Theorem 2.8, page 77 of [2]. Applying (11) to the ARX model $z_t = (-1)^n y_t^n = (\Pi_n \boldsymbol{h})^T (\Pi_n \nu_n(\boldsymbol{x}_t)) = \sum_{j=0}^{M_n(K)-2} h_j w_{t-j}$ gives the result for SARX models. ■

Theorem 1 establishes a condition on the input/output data under which our recursive identification scheme converges. Notice that this condition is a natural generalization of the persistence of excitation condition (11) for a single ARX model. However, it is important to notice that, although the recursive identification scheme does not take into account the evolution of the discrete state, the condition for convergence (14) does impose constraints on the mode sequence. For instance, let $n > 1$, and assume that the SARX model (1) always stays in one of the discrete states. Then it is easy to see that (14) is violated, because $\sum_{t=j}^{j+S} \Pi_n \nu_n(\boldsymbol{x}_t) \nu_n(\boldsymbol{x}_t)^T \Pi_n^T$ is not full rank for any S. Indeed, since the rank of $\sum \boldsymbol{x}_t \boldsymbol{x}_t^T$ is bounded above by $K-1$, the rank of $\sum \nu_n(\boldsymbol{x}_t)\nu_n(\boldsymbol{x}_t)^T$ is bounded above by $M_n(K-1) < M_n(K) - 1$, for $n > 1$, as stated by the following basic fact from algebraic geometry.

Lemma 1: Let V be a subspace of $\mathbb{R}^K$ of dimension k. Then the dimension of the subspace of $\mathbb{R}^{M_n(K)}$ spanned by $\{\nu_n(v) : v \in V\}$ is $M_n(k)$.

Proof: Without loss of generality, we can write V as $V = \{v \in \mathbb{R}^K : v_{k+1} = \cdots = v_K = 0\}$. Then, the entries of $\nu_n(v)$ are reduced to the $M_n(k)$ monomials $v_1^{n_1} \cdots v_k^{n_k}$, where $n_1 + \cdots + n_k = n$, which form a basis for the space of homogeneous polynomials of degree n in k variables. ■

We are therefore interested in understanding what restriction condition (11) imposes on the mode sequence $\{\lambda_t\}$. The following theorem gives necessary conditions on the mode sequence for the persistence of excitation condition for SARX models (14) to hold.

Theorem 2: For $i = 1, \ldots, n$, let $\sigma_i \subset \{1, \ldots, n\}$ be any choice of $i \leq n$ integers from the set $\{1, \ldots, n\}$, and $\mathcal{T}_i = \{t \in \{j, j+1, \ldots, j+S\} : \lambda_t = i\}$ be the set of time instances that mode i is visited in the interval $\{j, \ldots, j+S\}$. If (14) holds for some $S \in \mathbb{N}$ and $\rho_1, \rho_2 > 0$, then for all $i = 1, \ldots, n$ there exist $\rho_3, \rho_4 > 0$ such that for all $j \geq \max\{n_a, n_c\}$

$$\rho_3 I_{M_i(K)-1} \prec \sum_{t \in \cup_{k\in\sigma_i} \mathcal{T}_k} \Pi_i \nu_i(\boldsymbol{x}_t) \nu_i(\boldsymbol{x}_t)^T \Pi_i^T \prec \rho_4 I_{M_i(K)-1}. \tag{15}$$

Furthermore, the mode sequence λ_t satisfies

$$\forall\, i = 1, \ldots, n \quad \tau_i = \sum_{k\in\sigma_i} |\mathcal{T}_k| \geq M_i(K) - 1. \tag{16}$$

Proof: It is clear that (15) implies (16), because the matrix $\Gamma_i \in \mathbb{R}^{(M_i(K)-1)\times\tau_i}$, whose columns are $\Pi_i\nu_i(\boldsymbol{x}_t)$ for $t \in \cup_{k\in\sigma_i}\mathcal{T}_k$, must be full rank. Therefore, it is enough to prove (15), which we do by contradiction.

Assume first that there is an i such that Γ_i is not full rank. Then there is a nonzero vector $\boldsymbol{c}_i \in \mathbb{R}^{M_i(K)}$ whose last entry is zero such that $\boldsymbol{c}_i^T\nu_i(\boldsymbol{x}_t) = 0$ for all $t \in \cup_{k\in\sigma_i}\mathcal{T}_k$. Since in addition the regressors $\{\boldsymbol{x}_t\}_{t\in\cup_{k\in\sigma_i}\mathcal{T}_k}$ live in a union of i hyperplanes of $\mathbb{R}^K$, they must also satisfy $\boldsymbol{h}_i^T\nu_i(\boldsymbol{x}_t) = \prod_{k\in\sigma_i}(\boldsymbol{b}_k^T\boldsymbol{x}_t) = 0$, where the last entry of $\boldsymbol{h}_i \in \mathbb{R}^{M_i(K)}$ is equal to one, so that $\boldsymbol{h}_i \neq \boldsymbol{c}_i$. Therefore, there are two vectors $\boldsymbol{h} \neq \boldsymbol{c} \in \mathbb{R}^{M_n(K)}$ defined by $\boldsymbol{h}^T\nu_n(\boldsymbol{z}) = \prod_{i=1}^n(\boldsymbol{b}_i^T\boldsymbol{z})$ and $\boldsymbol{c}^T\nu_n(\boldsymbol{z}) = (\boldsymbol{c}_i^T\nu_i(\boldsymbol{z}))\prod_{k\notin\sigma_i}(\boldsymbol{b}_k^T\boldsymbol{z})$ such that $\boldsymbol{h}^T\nu_n(\boldsymbol{x}_t) = \boldsymbol{c}^T\nu_n(\boldsymbol{x}_t) = 0$ for all $t \in \{j,\ldots,j+S\}$. This implies that the matrix Γ_n must be rank deficient, which is in contradiction with the left hand side of (14).

Assume now that for some i there is a nonzero vector $\boldsymbol{c}_i \in \mathbb{R}^{M_i(K)}$ whose last entry is zero such that for all $\rho_4 > 0$ and $S \in \mathbb{N}$ there is a $j \geq \max\{n_a, n_c\}$ such that $\rho_4\|\boldsymbol{c}_i\|^2 \leq \sum_{t\in\cup_{k\in\sigma_i}\mathcal{T}_k}(\boldsymbol{c}_i^T\nu_i(\boldsymbol{x}_t))^2$. Since in addition $(\boldsymbol{c}_i^T\nu_i(\boldsymbol{x}_t))^2 \leq \|\boldsymbol{c}_i\|^2\|\nu_i(\boldsymbol{x}_t)\|^2$, we have that for all $\rho_4 > 0$ and $S \in \mathbb{N}$ there is a $j \geq \max\{n_a, n_c\}$ such that

$$\rho_4 \leq \sum_{t\in\cup_{k\in\sigma_i}\mathcal{T}_k}\|\nu_i(\boldsymbol{x}_t)\|^2. \tag{17}$$

Notice also that for all $\ell \in \mathbb{N}$ we have

$$\begin{aligned}\|\boldsymbol{x}\|^{2\ell} = \left(\sum_{k=1}^K x_k^2\right)^\ell &= \sum_{\ell_1+\ell_2+\cdots+\ell_K=\ell} P_{\ell_1,\ldots,\ell_K} x_1^{2\ell_1}\cdots x_K^{2\ell_K}\\ &= \sum_{\ell_1+\ell_2+\cdots+\ell_K=j}\binom{\ell}{\ell_1}\cdots\binom{\ell}{\ell_K}x_1^{2\ell_1}\cdots x_K^{2\ell_K}.\end{aligned}$$

Since $P_{\ell_1,\ldots,\ell_K} \geq 1$, if we let $\ell = i$ and $\ell = n$ in the above equation we obtain $\|\nu_i(\boldsymbol{x})\|^{2n} \leq \|\boldsymbol{x}\|^{2in} \leq P_n^i\|\nu_n(\boldsymbol{x})\|^{2i}$, where $P_\ell = \max_{\ell_1,\ldots,\ell_K} P_{\ell_1,\ldots,\ell_K}$. Replacing in (17) yields

$$\begin{aligned}\rho_4^{n/i} \leq \left(\sum_{t\in\cup_{k\in\sigma_i}\mathcal{T}_k}\|\nu_i(\boldsymbol{x}_t)\|^2\right)^{n/i} &\leq \sum_{t\in\cup_{k\in\sigma_i}\mathcal{T}_k}\sqrt[i]{\|\nu_i(\boldsymbol{x}_t)\|^{2n}}\\ &\leq \sum_{t\in\cup_{k\in\sigma_i}\mathcal{T}_k} P_n\|\nu_n(\boldsymbol{x}_t)\|^2.\end{aligned}$$

If we let $\rho_2 = \rho_4^{n/i}P_n^{-1}$, then for any $\boldsymbol{c} \in \mathbb{R}^{M_n(K)}$ we have

$$\rho_2\|\boldsymbol{c}\|^2 \leq \sum_{t=j}^S(\boldsymbol{c}^T\nu_n(\boldsymbol{x}_t))^2, \tag{18}$$

which is in contradiction with the right hand side of (14). ■

Loosely speaking, condition (16) of Theorem 2 states that if the persistence of excitation condition for SARX models (14) holds, then the mode sequence should visit any subset of the set of n modes a minimum number of times in any moving time window. This motivates the following definition of persistence of excitation for the mode sequence.

Definition 2 (Persistently exciting mode sequences): A mode sequence $\{\lambda_t\}$ is called *persistently exciting* if there is an $S \in \mathbb{N}$ such that for all $j \geq \max\{n_a, n_c\}$ (16) holds.

We are now interested in determining whether this persistence of excitation condition on the mode sequence together with the persistence of excitation condition on the input/output data generated by each ARX model (11) are sufficient to guarantee the convergence of our recursive identification algorithm. More specifically, is it true that if the output $\{y_t\}$ is bounded, the mode sequence $\{\lambda_t\}$ is persistently exciting, and there exists an $S \in \mathbb{N}$ and $\rho_1, \rho_2 > 0$ such that for all $j \in \mathbb{N}$ and for all $i = 1, \ldots, n$

$$\rho_1 I_{K-1} \prec \sum_{t\in\mathcal{T}_i}\Pi_1\boldsymbol{x}_t\boldsymbol{x}_t^T\Pi_1^T \prec \rho_2 I_{K-1}, \tag{19}$$

where $\mathcal{T}_i = \{t \in \{j, j+1, \cdots, j+S\} : \lambda_t = i\}$, then $\boldsymbol{h} - \widehat{\boldsymbol{h}}_t$ converges exponentially to zero?

Unfortunately, the answer is no. To see this, consider a second order SARX model

$$y_t = c_1(\lambda_t)u_{t-1} + c_2(\lambda_t)u_{t-2}$$

with a periodic mode sequence defined for $t \geq 2$

$$\lambda_t = \{1,1,1,1|2,2,2,2|1,1,1,1|2,2,2,2|\cdots\}$$

and a periodic (for $t \geq 1$) input sequence defined for $t \geq 0$

$$u_t = \{1|1,1,1,1|-1,1,-1,1|1,1,1,1|-1,1,-1,1|\cdots\}.$$

Since $n = 2$ and $K = 3$ we have $M_1(K) = 3$ and $M_2(K) = 6$. Therefore, the mode sequence is persistently exciting with $S = 5$, because each mode is visited al least 2 times and both modes are visited at least 5 times in any moving window of size at least 6. The input/output data is also persistently exciting with $S = 4$, because for $t \geq 2$ the sequence of $(M_1(K) - 1 = 2)$-dimensional regressors

$$\Pi_1\boldsymbol{x}_t = \left[\begin{array}{c|c|c|c} 1\;1\;1\;1 & 1\;-1\;\;\;1\;-1 & 1\;1\;1\;1 & \cdots\\ 1\;1\;1\;1 & -1\;\;\;1\;-1\;\;\;1 & 1\;1\;1\;1 & \cdots\end{array}\right]$$

is of rank 2 for any moving time window of size at least 5. However, the sequence of embedded regressors of dimension $M_2(K) - 1 = 5$ defined for $t \geq 2$ as

$$\Pi_2\nu_2(\boldsymbol{x}_t) = \left[\begin{array}{c|c|c|c} 1\;1\;1\;1 & 1\;\;\;1\;\;\;1\;\;\;1 & 1\;1\;1\;1 & \cdots\\ 1\;1\;1\;1 & -1\;-1\;-1\;-1 & 1\;1\;1\;1 & \cdots\\ r\;r\;r\;r & s\;\;\;s\;\;\;s\;\;\;s & r\;r\;r\;r & \cdots\\ 1\;1\;1\;1 & 1\;\;\;1\;\;\;1\;\;\;1 & 1\;1\;1\;1 & \cdots\\ r\;r\;r\;r & -s\;-s\;-s\;-s & r\;r\;r\;r & \cdots\end{array}\right],$$

where $r = c_1(1) + c_2(2)$ and $s = c_1(1) - c_2(2)$, is of rank at most 2 for any moving window of any size. This is because the input/output data, although persistently exciting, satisfies *e.g.* the polynomial of degree two $u_{t-1}^2 - u_{t-2}^2 = 0$.

The above example suggests that the main difficulty that prevents us from decoupling the persistence of excitation condition on the input/output data generated by an SARX model (14) into n persistence of excitation conditions on the mode sequence and the input/output data generated by the individual ARX models is that the input/output data can potentially satisfy a homogeneous polynomial of degree n other than the polynomial $\boldsymbol{h}^T\nu_n(\boldsymbol{z})$ defining the SARX model. Although we could impose as a condition on the input/output data that no polynomial of degree n other than $\boldsymbol{h}^T\nu_n(\boldsymbol{z})$ fits the data, such a condition would essentially be the same as assuming that (14) holds.

It remains an open issue whether it is possible to achieve exponential convergence by imposing persistence of excitation conditions on the input and mode sequences only, thus generalizing the persistence of excitation condition on the input sequence (12) from ARX to SARX models.

V. Recursive Identification of the Parameters of the Individual ARX Models

The recursive identification algorithm proposed in the previous section provides an estimate $\hat{\boldsymbol{h}}_t$ of the hybrid model parameters given the data up to time t. Under suitable conditions, such an estimate converges exponentially to $\boldsymbol{h}$ as $t \to \infty$. The rest of the problem is to obtain an estimate $\hat{\boldsymbol{b}}_{\lambda_t}$ of the model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$ at time t from $\hat{\boldsymbol{h}}_t$.

To this end, recall from [15], [18], [19] that given $\boldsymbol{h}$ one may recover the model parameters generating the regressor vector $\boldsymbol{x}_t$ by looking at the partial derivative of the hybrid decoupling polynomial $p_n(\boldsymbol{z}) = \boldsymbol{h}^T\nu_n(\boldsymbol{z})$ at $\boldsymbol{z} = \boldsymbol{x}_t$

$$Dp_n(\boldsymbol{x}_t) = \frac{\partial p_n(\boldsymbol{x}_t)}{\partial \boldsymbol{x}_t} = \sum_{i=1}^{n} \prod_{\ell \neq i} (\boldsymbol{b}_\ell^T \boldsymbol{x}_t) \boldsymbol{b}_i. \tag{20}$$

This is because if the current measurement $\boldsymbol{x}_t$ is generated by the i^{th} ARX model, all the terms in the above summation vanish, except for the ith, and so $\boldsymbol{b}_i \sim Dp_n(\boldsymbol{x}_t)$. Since in addition the K^{th} entry of $\boldsymbol{b}_i$ is equal to one, we obtain

$$\boldsymbol{b}_i = \frac{Dp_n(\boldsymbol{x}_t)}{e_K^T Dp_n(\boldsymbol{x}_t)}, \tag{21}$$

where $e_K = [0, \cdots, 0, 1]^T \in \mathbb{R}^K$. Therefore, given an estimate $\hat{\boldsymbol{h}}_t$ of the hybrid model parameters up to time t, we may identify the parameters of the current ARX model as

$$\hat{\boldsymbol{b}}_{\lambda_t} = \frac{D\nu_n^T(\boldsymbol{x}_t)\hat{\boldsymbol{h}}_t}{e_K^T D\nu_n^T(\boldsymbol{x}_t)\hat{\boldsymbol{h}}_t}. \tag{22}$$

Since the estimates of the hybrid model parameters converge to $\boldsymbol{h}$, and $\boldsymbol{h}$ is uniquely determined by the parameters of the ARX models, $\{\boldsymbol{b}_i\}_{i=1}^n$, the estimates of parameters of each ARX model also converge, in spite of the switching among different ARX models, as stated by the following theorem.

Theorem 3: Consider a minimal SARX system of the form (1) and assume that the recursive identification scheme defined by (13) and (22) is used. If there exist $\rho_1, \rho_2 > 0$ and an integer S such that for all $j \geq \max\{n_a, n_c\}$

$$\rho_1 I_{M_n(K)-1} \prec \sum_{t=j}^{j+S} \Pi_n \nu_n(\boldsymbol{x}_t)\nu_n^T(\boldsymbol{x}_t)\Pi_n^T \prec \rho_2 I_{M_n(K)-1}, \tag{23}$$

then $\boldsymbol{b}_{\lambda_t} - \hat{\boldsymbol{b}}_{\lambda_t}$ converges to zero exponentially.

Proof: By assumption, the SARX model is minimal. Therefore, there is a T such that for all $t \geq T$ there is a unique i such that $\boldsymbol{b}_i^T \boldsymbol{x}_t = 0$. This implies that the polynomial $\boldsymbol{h}^T\nu_n(\boldsymbol{z})$ has no repeated factor, hence there is a $\delta > 0$ such that $\|D\nu_n(\boldsymbol{x}_t)^T\boldsymbol{h}\| \geq \delta$. From Theorem 1 we have that $\hat{\boldsymbol{h}}_t - \boldsymbol{h}$ converges to zero exponentially, *i.e.* there are $\kappa, \lambda > 0$ such that $\|\hat{\boldsymbol{h}}_t - \boldsymbol{h}\| < \kappa\lambda^{-t}$. Therefore, $\|D\nu_n(\boldsymbol{x}_t)^T\hat{\boldsymbol{h}}_t\| \geq \delta$, else there is a sequence of times $t_1 < t_2 < \cdots$ such that $\delta < \|D\nu_n(\boldsymbol{x}_{t_j})^T\boldsymbol{h}\| = \|D\nu_n(\boldsymbol{x}_{t_j})^T(\boldsymbol{h} - \hat{\boldsymbol{h}}_{t_j} + \hat{\boldsymbol{h}}_{t_j})\| \leq \|E_n\|\rho_2\kappa\lambda^{-t_j} + \delta$, where $E_n \in \mathbb{R}^{M_n(K)\times M_{n-1}(K)}$ is a constant matrix containing the exponents of the Veronese map of degree n and is such that $D\nu_n(\boldsymbol{x}_t) = E_n\nu_{n-1}(\boldsymbol{x}_t)$. Therefore,

$$\begin{aligned}
&\|\boldsymbol{b}_{\lambda_t} - \hat{\boldsymbol{b}}_{\lambda_t}\| = \\
&\left\|\frac{e_K^T D\nu_n^T(\boldsymbol{x}_t)\hat{\boldsymbol{h}}_t D\nu_n^T(\boldsymbol{x}_t)\boldsymbol{h} - e_K^T D\nu_n^T(\boldsymbol{x}_t)\boldsymbol{h} D\nu_n^T(\boldsymbol{x}_t)\hat{\boldsymbol{h}}_t}{|e_K^T D\nu_n^T(\boldsymbol{x}_t)\hat{\boldsymbol{h}}_t||e_K^T D\nu_n^T(\boldsymbol{x}_t)\boldsymbol{h}|}\right\| = \\
&\left\|\frac{e_K^T D\nu_n^T(\boldsymbol{x}_t)(\hat{\boldsymbol{h}}_t - \boldsymbol{h}) D\nu_n^T(\boldsymbol{x}_t)\boldsymbol{h} - e_K^T D\nu_n^T(\boldsymbol{x}_t)\boldsymbol{h} D\nu_n^T(\boldsymbol{x}_t)(\hat{\boldsymbol{h}}_t - \boldsymbol{h})}{|e_K^T D\nu_n^T(\boldsymbol{x}_t)\hat{\boldsymbol{h}}_t||e_K^T D\nu_n^T(\boldsymbol{x}_t)\boldsymbol{h}|}\right\| \\
&\leq 2\frac{\|D\nu_n^T(\boldsymbol{x}_t)(\hat{\boldsymbol{h}}_t - \boldsymbol{h})\|\|D\nu_n^T(\boldsymbol{x}_t)\boldsymbol{h}\|}{|e_K^T D\nu_n^T(\boldsymbol{x}_t)\hat{\boldsymbol{h}}_t||e_K^T D\nu_n^T(\boldsymbol{x}_t)\boldsymbol{h}|} \leq 2\frac{\rho_2^2\|E_n\|^2\|\boldsymbol{h}\|\kappa\lambda^{-t}}{\delta^2}.
\end{aligned}$$

■

VI. Experiments

In this section we present experiments evaluating the performance of the proposed algorithm as a function of the mode sequence and the level of noise. We consider a first order SARX model

$$y_t = a(\lambda_t)y_{t-1} + b(\lambda_t)u_{t-1} + w_{t-1} \tag{24}$$

with mode $\lambda_t \in \{0, 1\}$, input $u_t \sim \mathcal{N}(0, 1)$, noise $w_t \sim \mathcal{N}(0, \sigma^2)$, and parameters $a(1) = -0.9$, $a(2) = 0.7$, $c(1) = 1$ and $c(2) = -1$, so that $\boldsymbol{h} = [-1, 1.6, 0, -0.63, 0.2, 1]^T \in \mathbb{R}^6$. We set the parameter of the recursive identifier to $\mu = 1$.

Figure 1 shows the evolution of the model parameters in the absence of noise starting from $\Pi_2\hat{\boldsymbol{h}}_0 = 0$. The top and middle figures show that the estimated parameters converge to their true values in approximately 50 and 100 seconds, respectively, when λ_t alternates periodically between the two modes with a period of 2 and 30 seconds, respectively. This suggests that the speed of convergence of the algorithm depends on the speed of the mode sequence: the faster the mode sequence the faster $\hat{\boldsymbol{h}}_t$ converges. The bottom figure shows that even though the hybrid model parameters have not converged to their true values for $t \in [0, 200]$ (they converge after 800 seconds), the ARX model parameters approach $a(1), c(1)$ and $a(2), c(2)$ after approximately 20 and 130 seconds, respectively. This suggests that when λ_t stays constant for a long period of time one may still be able to estimate the ARX model parameters correctly, even though the hybrid model parameters have not yet converged.

Figure 2 shows the evolution of the model parameters starting from $\Pi_2\hat{\boldsymbol{h}}_0 = 0$ for a noise level of $\sigma = 0.02$ (top) and $\sigma = 0.05$ (bottom). Even though the algorithm is designed for noiseless measurements, it also works in the presence of noise. However, the performance of the algorithm deteriorates as σ increases, as expected.

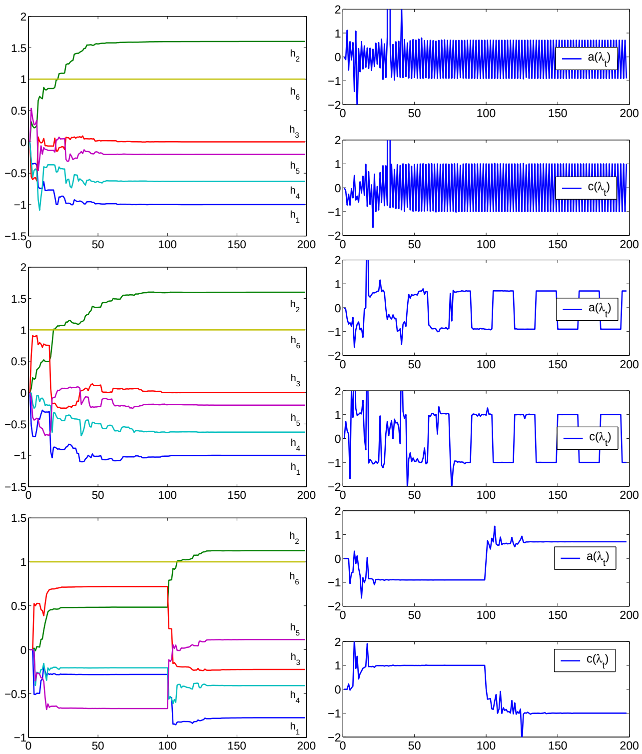

Fig. 1. Evolution of $\hat{\boldsymbol{h}}_t$ (left), $\hat{a}(\lambda_t)$ and $\hat{c}(\lambda_t)$ (right) for $\Pi_2 \hat{\boldsymbol{h}}_0 = 0$, $\sigma = 0$ and λ_t of period of 2 (top), 30 (middle) and 200 seconds (bottom).

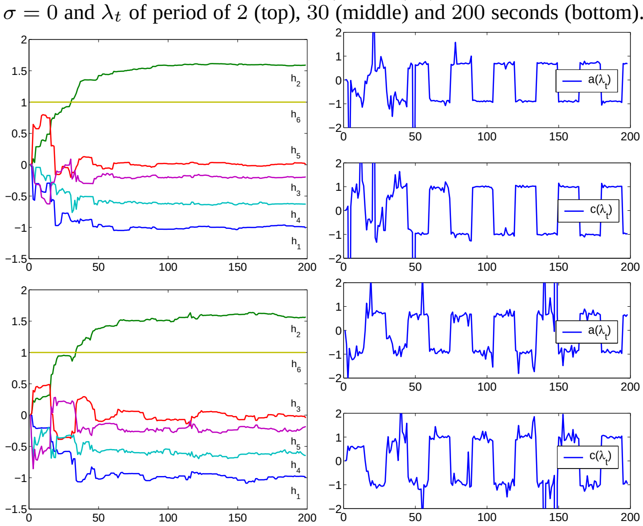

Fig. 2. Evolution of $\hat{\boldsymbol{h}}_t$ (left), $\hat{a}(\lambda_t)$ and $\hat{c}(\lambda_t)$ (right) for $\Pi_2 \hat{\boldsymbol{h}}_0 = 0$, λ_t of period 30 seconds, $\sigma = 0.02$ (top) and $\sigma = 0.05$ (bottom).

VII. Conclusions and Future Work

We have presented a recursive algorithm for identifying the parameters of switched ARX models with known and equal orders and derived persistence of excitation conditions on the input/output data that guarantee the exponential convergence of the identifier. It should be easy to extend our approach to systems of unknown and different orders, by combining our results with the batch algorithm of [15] for that case. It remains open to determine persistence of excitation conditions on the input and mode sequences only, and to extend the algorithm to multivariate SARX models.

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