Large Deformations and Triangulation for Image Matching Problems

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Objective and Context :

Given 2 images, I_0 a template and I_1 a target, we seek a function ϕ with smoothness properties such as : $\phi(I_0) \simeq I_1$

Our framework is the Deformable Template model (Grenander) in particular the Large Deformation Theory (Miller-Trouvé-Younes)

Mathematical backgroud and Notations :

Large Deformations

* Let $\Omega \in \mathbb{R}^d$ be an open set. The framework defines a class of **deformations** $\phi : \Omega \to \Omega$, **objects** (eg : images $I : \Omega \to \mathbb{R}$, landmarks $(x_i)_{1\leq i\leq N}, \ldots$) and a specific **groupe action** (eg : $\phi.I = I \circ \phi^{-1}$, $\phi.((x_i)_{1\leq i\leq N}) = (\phi.(x_i))_{1\leq i\leq N}$)
Deformations ϕ are built by integrating time-dependent vector field $v_t : \Omega \to \mathbb{R}^l$ (in our cases $l = d = 2, 3$) : $\phi = \phi_1^v$

$$\left\{ \frac{d\phi_t^v}{dt} = v_t \circ \phi_t^v \quad \text{and} \quad \phi_0 = Id \right.$$

* $v_t \in V$ Hilbert space of regular vector field, in particular, V is a functional space, continuously embedded in $C^0(\Omega)$, defined by an operateur L and its **reproducing kernel** $K_V = L^{-1}$ such as :

$$\forall (v, w) \in V^2,\ \langle v, w\rangle_V = \langle K_V v, w\rangle_{L^2(\Omega)}$$

* $\{\phi_1^v, v \in L^2([0,1], V)\}$ is a **subgroup of diffeomorphisms** on Ω, equipped with a right-invariant metric : $d(\phi, \psi) = d(Id, \psi \circ \phi^{-1})$.

* The **distance** between two objects O_0 and O_1 is computed via the group action :

$$
\begin{aligned}
d(O_0, O_1) &= \inf_{v_t \in V, \phi_1^v(O_0) = O_1} d(Id, \phi_1^v) \\
&= \inf_{v_t \in V, \phi_1^v(O_0) = O_1} \{\textstyle\int_0^1 ||v_t||_V^2 dt\}
\end{aligned}
$$

Method to find the velocity vector field

We seek v_t by minimizing an energy that takes into account two terms : the path length (given by the previous formula) and a measurement of the difference between our data :

$$v_t = \arg\min\{\frac{1}{2}\int_0^1 ||v_t||_V^2 dt + \lambda g(O_0, O_1, \phi^v)\}$$

2 different approaches

Image matching (Beg) Given 2 images I_0 and I_1, v_t is determined everywhere on the domain for every time t by minimizing :

$$E = \frac{1}{2}\int_0^1 ||v_t||_V^2 dt + \lambda \int_\Omega |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy$$

Landmark Matching (Joshi - Miller) Given several template landmarks $(x_i)_{1\leq i\leq N}$ and target landmarks $(y_i)_{1\leq i\leq N}$, v_t is determined by minimizing the following energy :

$$E = \frac{1}{2}\int_0^1 ||v_t||_V^2 dt + \lambda \sum_{i=1}^{N} ||\phi_1^v(x_i) - y_i||_{\mathbb{R}^k}^2.$$

that in fact depends only on a new variable :
$(p_i(t) = L(v_i(t)))_{1\leq i\leq N}$, momentum of the deformation at time t.
To reconstruct v_t on the whole domain, we use the **interpolation formula** :

$$\forall x \in \Omega,\ v_t(x) = \sum_{i=1}^{N} K(x_i(t), x) p_i(t)$$

Our approach is a combination between these 2 points of view

Our model :

Data 2 images, a template I_0, a target I_1 and landmarks chosen on the template $(x_i)_{1\leq i\leq N}$.

Wanted ϕ_v that matches the template on the target but only dependent on the landmark set.

Idea : A **triangulation** of the template,

An **affine transformation on each triangle and continuous on the whole domain**.

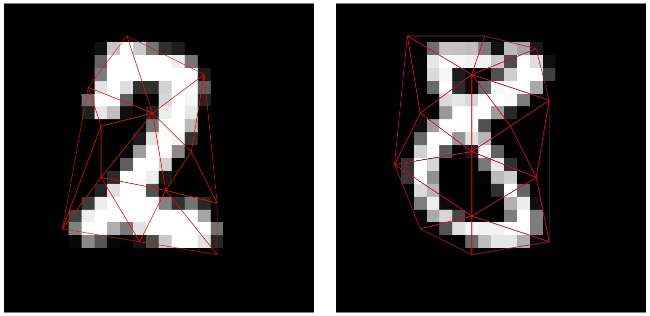
This yields that the deformation is only determined by the **vertices** of the triangulation (our landmarks)

Resulting Energy :

$$E = \frac{1}{2}\int_0^1 ||v_t||_V^2 dt + \lambda \sum_{i=1}^{r} \int_{\phi(T_i)} |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy$$

Triangulation examples

We use the **Delaunay's triangulation** of a point set :



Problem Reformulation :

Conservation of Momentum Property

Using the "conservation of Momentum" property of the geodesics (Miller - Trouvé - Younes) the momentum at time t is determined by the momentum configuration at time $t = 0$: **Momentum Evolution Equation** : $p_t(x) = Lv_t(x) = Lv_0((d\phi_t)^{-1}x \circ \phi_t)$

and the **Euler's equation** of geodesics : $p_1(p_0) + \lambda \nabla_x g = 0$

Consequence : equation for geodesic evolution depends only on the template and the momentum at time 0

So p_0 is an appropriate variable of the problem.

Hamiltonian framework

Evolution equations describing the transport of the template along the geodesics (cf : M.I.Miller A.Trouvé L. Younes) : let q be the point and p the momentum : **Hamilton's Equations**

$$\begin{cases} \frac{dq_i(t)}{dt} & = & \sum_{j=1}^{N} K(q_j(t), q_i(t))p_j(t) = K(q_i(t))p(t) \\ \frac{dp_i(t)}{dt} & = & -(d_{q_i(t)} v_t)^* p_i(t) \end{cases}$$

With these equations, we search the best initial conditions that give rise to the minimizing trajectory.

The path length term can be rewritten :

$$d = \frac{1}{2}\int_0^1 ||v_t||_V^2 dt = \frac{1}{2}\int_0^1 \frac{dq}{dt}^* K(q(t))^{-1} \frac{dq}{dt} dt$$

therefore $\frac{dq}{dt} = K(q(t))p(t)$ then : $d = \frac{1}{2}\int_0^1 p(t)^* K(q(t))p(t)dt$ where $H(q(t),p(t)) = \frac{1}{2}p(t)^* K(q(t))p(t)$ is known as the Hamiltonian.

Property of the Hamiltonian : $H(q(t),p(t))$ is a constant function of time.

So we finally have the **Hamiltonian system** :

$$\begin{cases} \frac{dq(t)}{dt} & = & K_\sigma p \\ \frac{dp(t)}{dt} & = & \frac{1}{2}\langle K_\sigma p, p\rangle \end{cases}$$

the **Euler's equation :** $p_1(p_0) + \lambda \nabla_x g = 0$

And the energy to minimize is :

$$E = \frac{1}{2} p(0)^* K(q(0)) p(0) + \lambda \sum_{i=1}^{r} \int_{\phi(T_i)} |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy$$

with respect to p_0.

Algorithms :

Gradient descent

The gradient descent is computed in the initial momentum space (cf : M. Vaillant M.I. Miller L. Younes A. Trouvé).

Energy to minimize :

$$E = \frac{1}{2} p(0)^* K(q(0)) p(0) + \lambda \sum_{i=1}^{r} \int_{\phi(T_i)} |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy$$

Algorithm :

Let $g(x)$ be the data attachment term, and $q_i^1 = \phi_1(x_i)$.

$$\begin{aligned} p_0^{k+1} &= p_0^k - \alpha \nabla_{p_0} E \\ &= p_0^k - \alpha (K(q_0) p_0 + \lambda \tfrac{dg}{dq^1} \tfrac{dq^1}{dp_0}) \end{aligned}$$

Newton's method

We solve the Euler's Equation :

$$G(p_0) = p_1(p_0) + \lambda \nabla_{q^1} g(q^1(p_0)) = 0$$

Algorithm :

$$\begin{aligned} p_0^{k+1} &= p_0^k - (d_{p_0} G)^{-1}\ G(p_0^k) \\ &= p_0^k - (\tfrac{dp_1}{dp_0} + \lambda \tfrac{d^2 g}{(dq^1)^2} \tfrac{dq^1}{dp_0})^{-1} (p_1(p_0) + \lambda \nabla_x g(q_1(p_0))) \end{aligned}$$

Gradient Computation

The gradient and the second derivative of g are needed. $g(q^1) = \sum_{i=1}^{r} \int_{\phi(T_i)} |I_0 \circ \phi^{-1}(y) - I_1(y)|^2 dy = \sum_{i=1}^{r} \int_{T_i} |I_0(x) - I_1(\phi(x))|^2 |d_x\phi| dx$ As ϕ is sought affine by part we can use the **barycentric coordinates** :

Let S be the ideal simplex $((0,0),(1,0),(0,1))$ and M_i, M_i' be the 2 linear applications :

$$\left\{ \begin{array}{ccccc} S & \mapsto & M_i'(S) & = & \phi_v(T_i) \\ S & \mapsto & M_i(S) & = & T_i \end{array} \right.$$

And $q_{\varepsilon,i}(\alpha,\beta) = x^1_{\varepsilon,i} + \alpha(x^2_{\varepsilon,i} - x^1_{\varepsilon,i}) + \beta(x^3_{\varepsilon,i} - x^1_{\varepsilon,i})$, for $\varepsilon = 0, 1$, and $1 \le i \le N$.

Then, $x = M_i(\alpha,\beta)$, $\phi(x) = M_i'(\alpha,\beta) = M_i'(M_i^{-1}(x))$ and $|d_x\phi| = |M_i' M_i^{-1}|$.

Thus :

$$g(\mathbf{z}) = \sum_{i=1}^{r} \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} |I_1(q_{1,i}(\alpha,\beta)) - I_0(q_{0,i}(\alpha,\beta))|^2 |M_i'| d\alpha d\beta$$

where $\mathbf{z} = (\phi_v(x_1), ..., \phi_v(x_N))^T$. Let $|A_i| = |M_i'|$.

And its gradient :

$$\frac{\partial g}{\partial \mathbf{z}} = \sum_{i=1}^{r} \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} 2(I_1(q_{1,i}(\alpha,\beta)) - I_0(q_{0,i}(\alpha,\beta)))|A_i(z_i)| \\ (\partial_{z_i} q_{1,i}(\alpha,\beta))^* \nabla I_1(q_{1,i}(\alpha,\beta)) d\alpha d\beta \\ + \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} |I_1(q_{1,i}(\alpha,\beta)) - I_0(q_{0,i}(\alpha,\beta))|^2 \partial_{z_i}(|A_i(z_i)|) d\alpha d\beta$$

Second derivative

We also get for a fixed triangle T_i : $\frac{\partial^2 g_i}{\partial \mathbf{z}^2} =$

$$\int_{\alpha=0}^{1}\int_{\beta=0}^{1-\alpha} 2(\delta z_2)^*(\delta_z q_{1,i})^*\nabla_{q_{1,i}} I_1(\nabla_{q_{1,i}} I_1)^*(\delta_z q_{1,i})\delta z_1|A_i(z_i)|d\alpha d\beta$$

$$+\int_{\alpha=0}^{1}\int_{\beta=0}^{1-\alpha} 2(\delta z_2)^*(\delta_z q_{1,i})^*(\partial^2_{q_{1,i},q_{1,i}} I_1)(\delta_z q_{1,i})\delta z_1|A_i(z_i)|$$

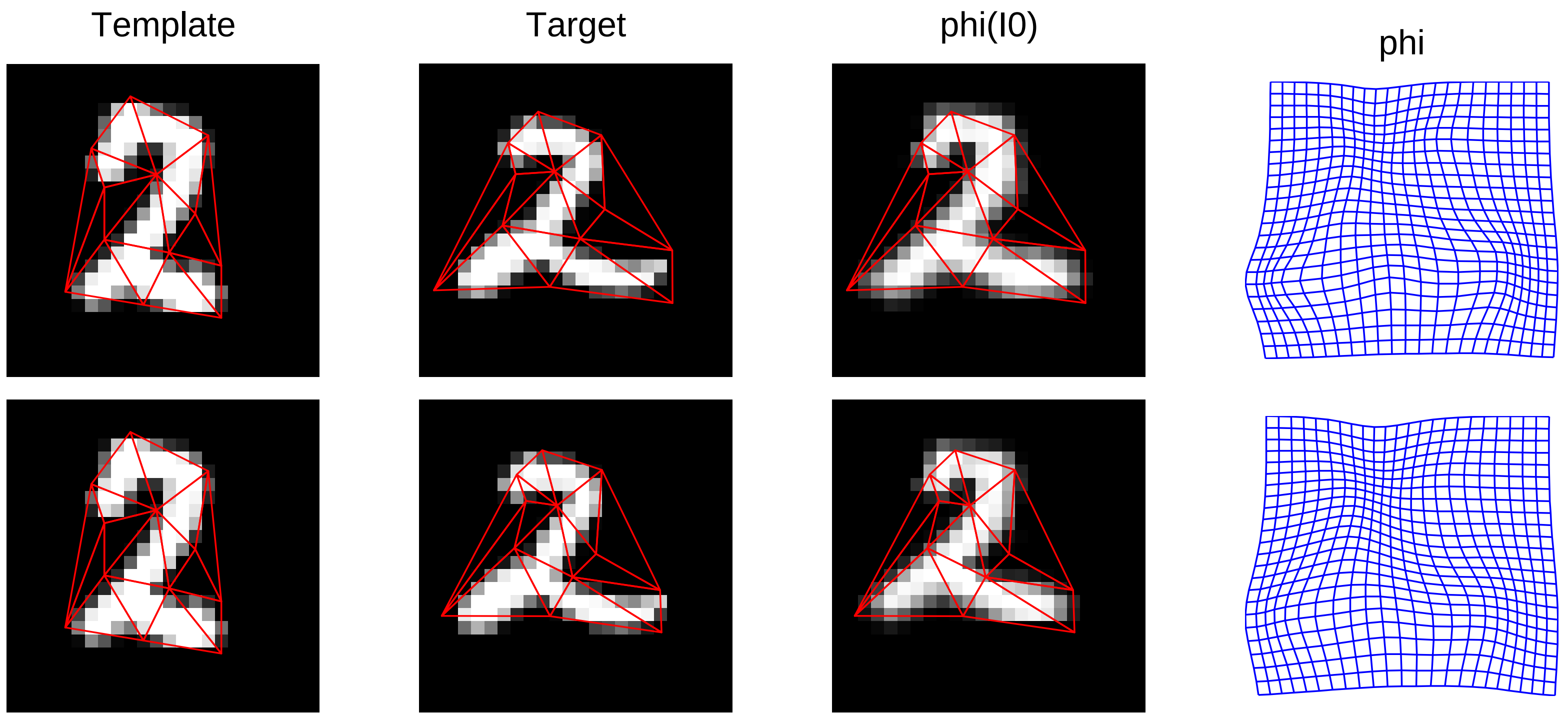
$$(I_1(q_{1,i}(\alpha,\beta)) - I_0(q_{0,i}(\alpha,\beta)))d\alpha d\beta$$

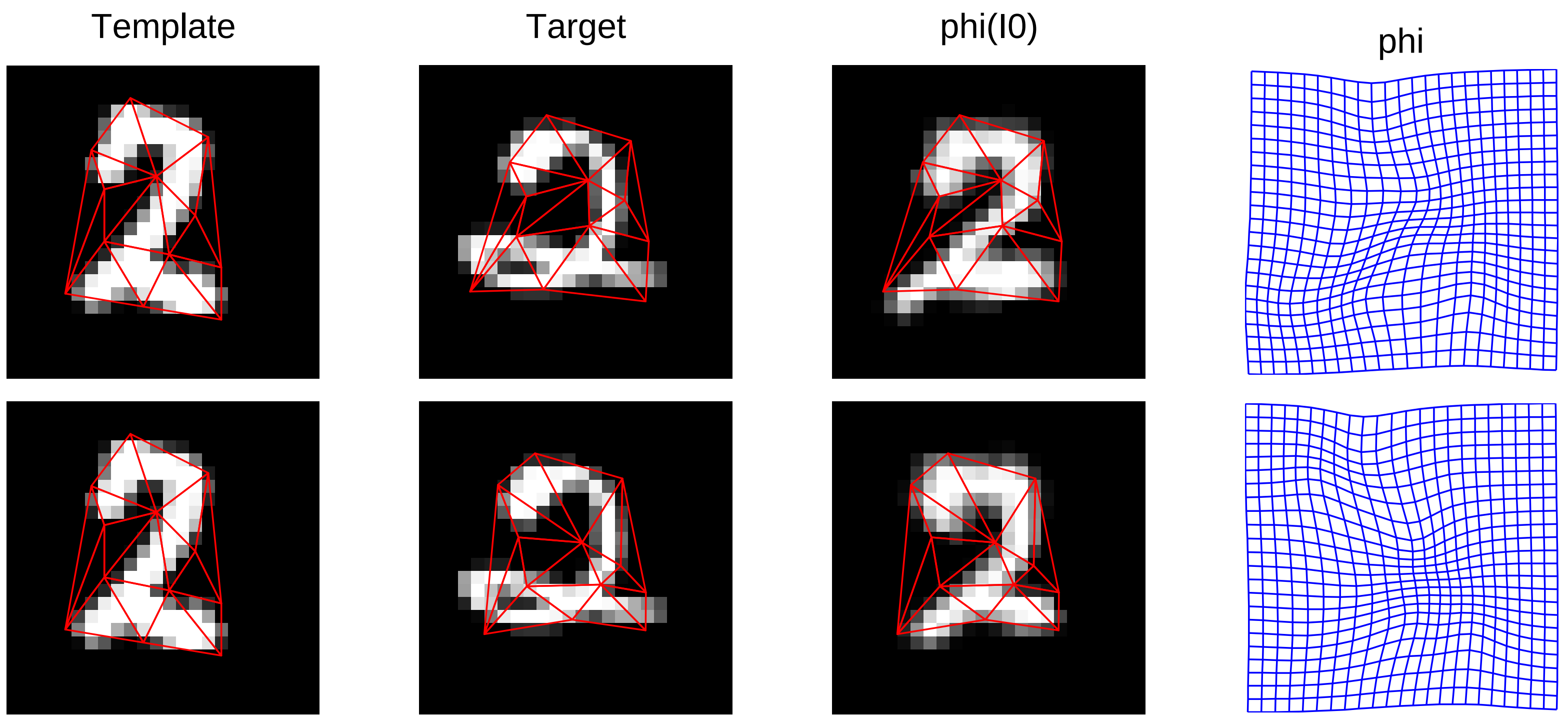
$$+\int_{\alpha=0}^{1}\int_{\beta=0}^{1-\alpha} 2(I_1(q_{1,i}(\alpha,\beta)) - I_0(q_{0,i}(\alpha,\beta)))(\delta z_2)^*(\delta_z q_{1,i})^*$$

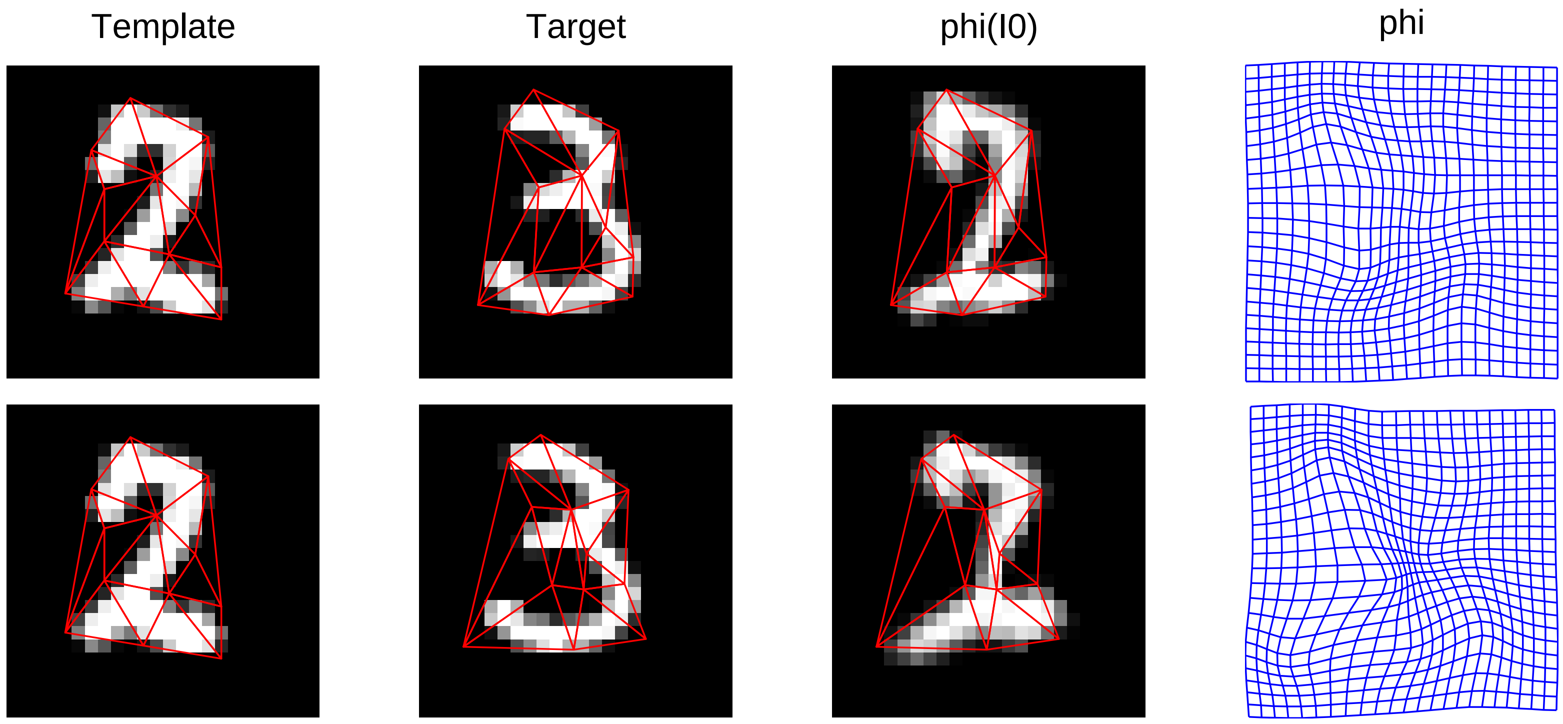
$$\nabla_{q_{1,i}} I_1(\nabla_z|A_i(z_i)|)^*\delta z_1 d\alpha d\beta$$

$$+\int_{\alpha=0}^{1}\int_{\beta=0}^{1-\alpha}(I_1(q_{1,i}(\alpha,\beta)) - I_0(q_{0,i}(\alpha,\beta))^2(\delta z_2)^*\partial^2_{z,z}|A_i(z_i)|\delta z_1 d\alpha d\beta]$$

Experiments




Template
Target
phi(I0)
phi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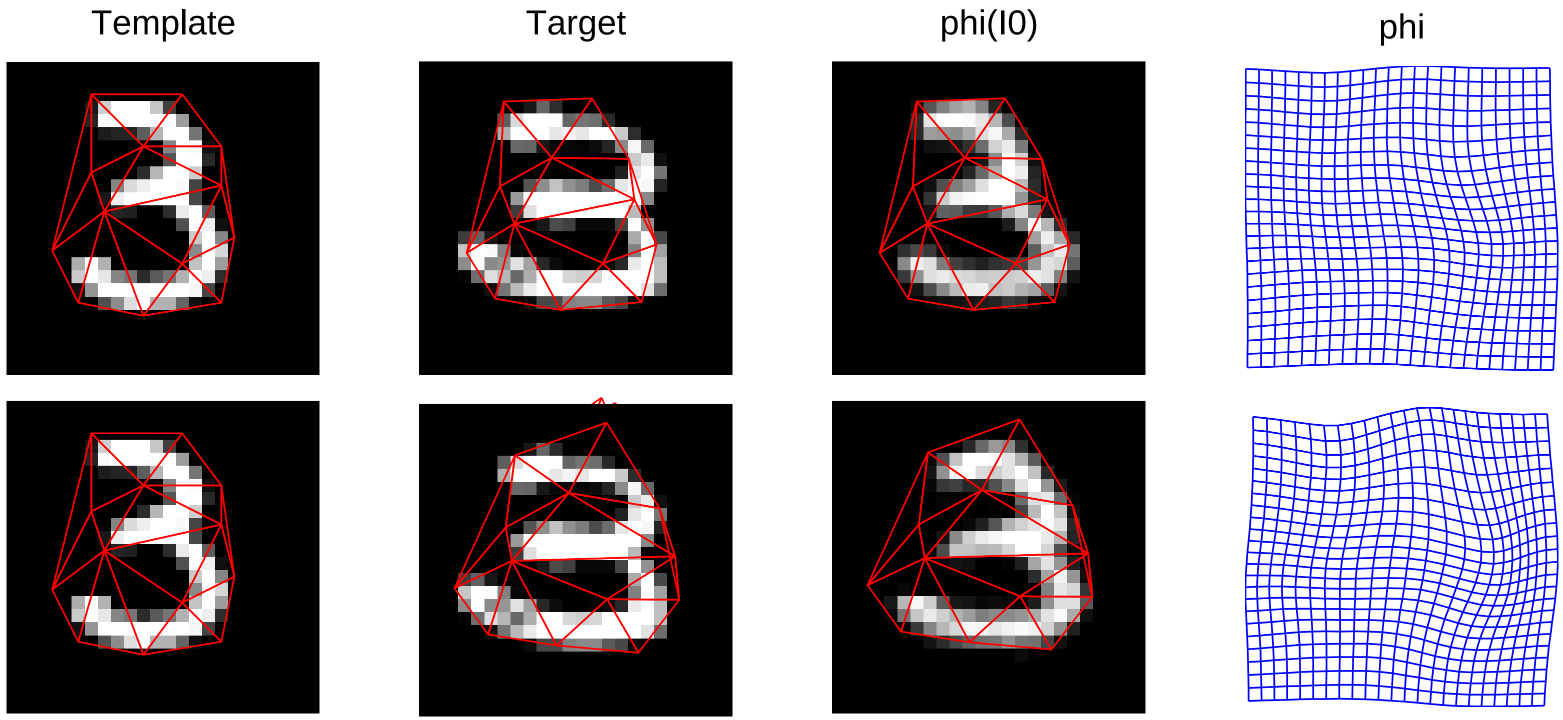
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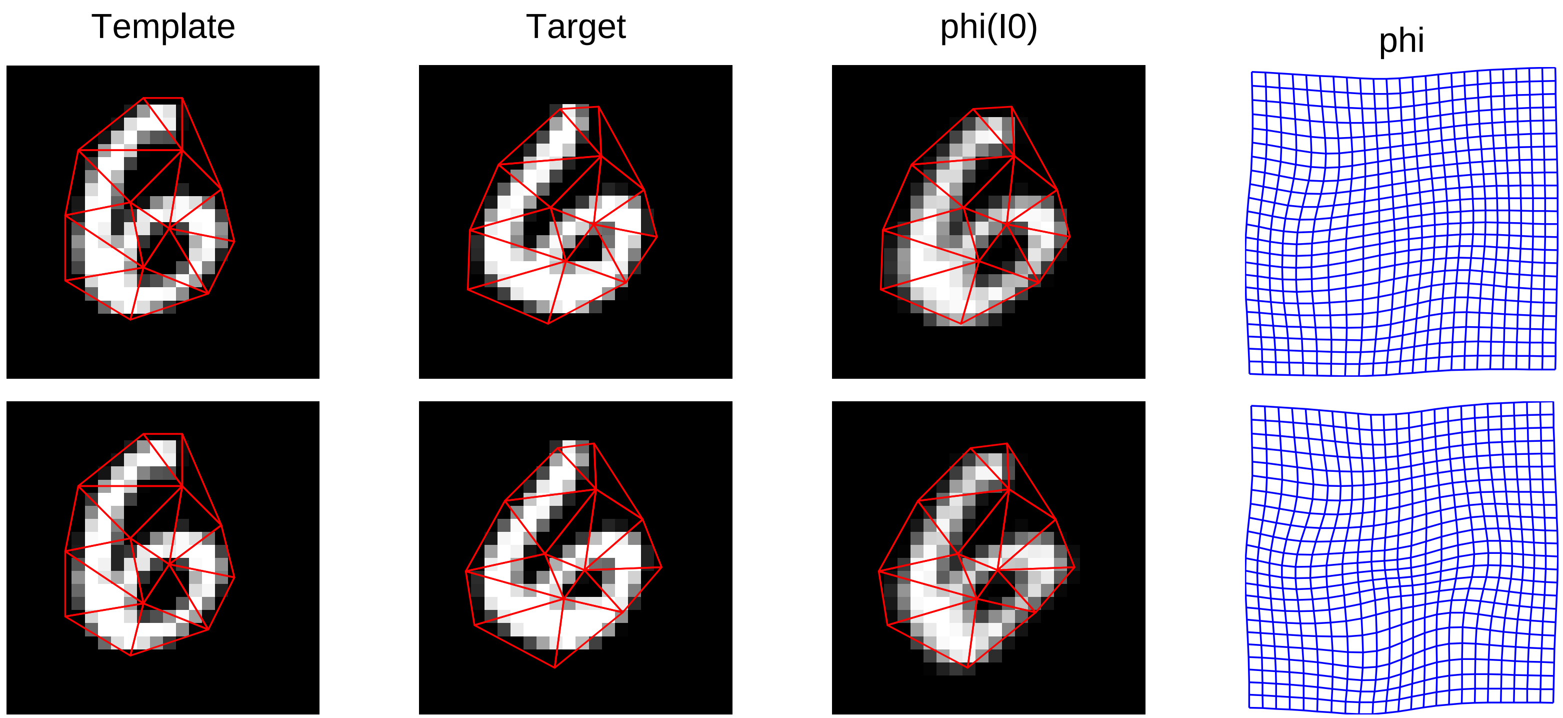
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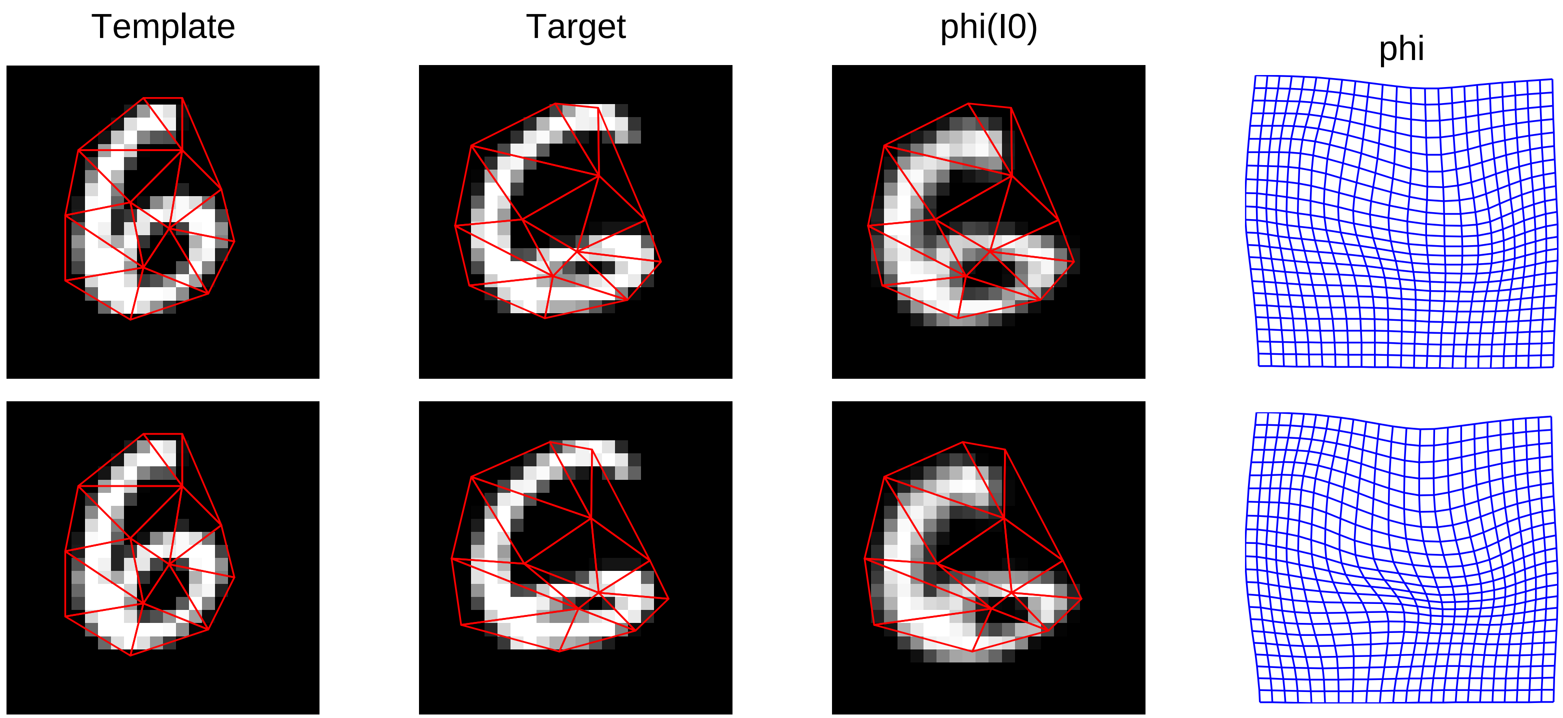
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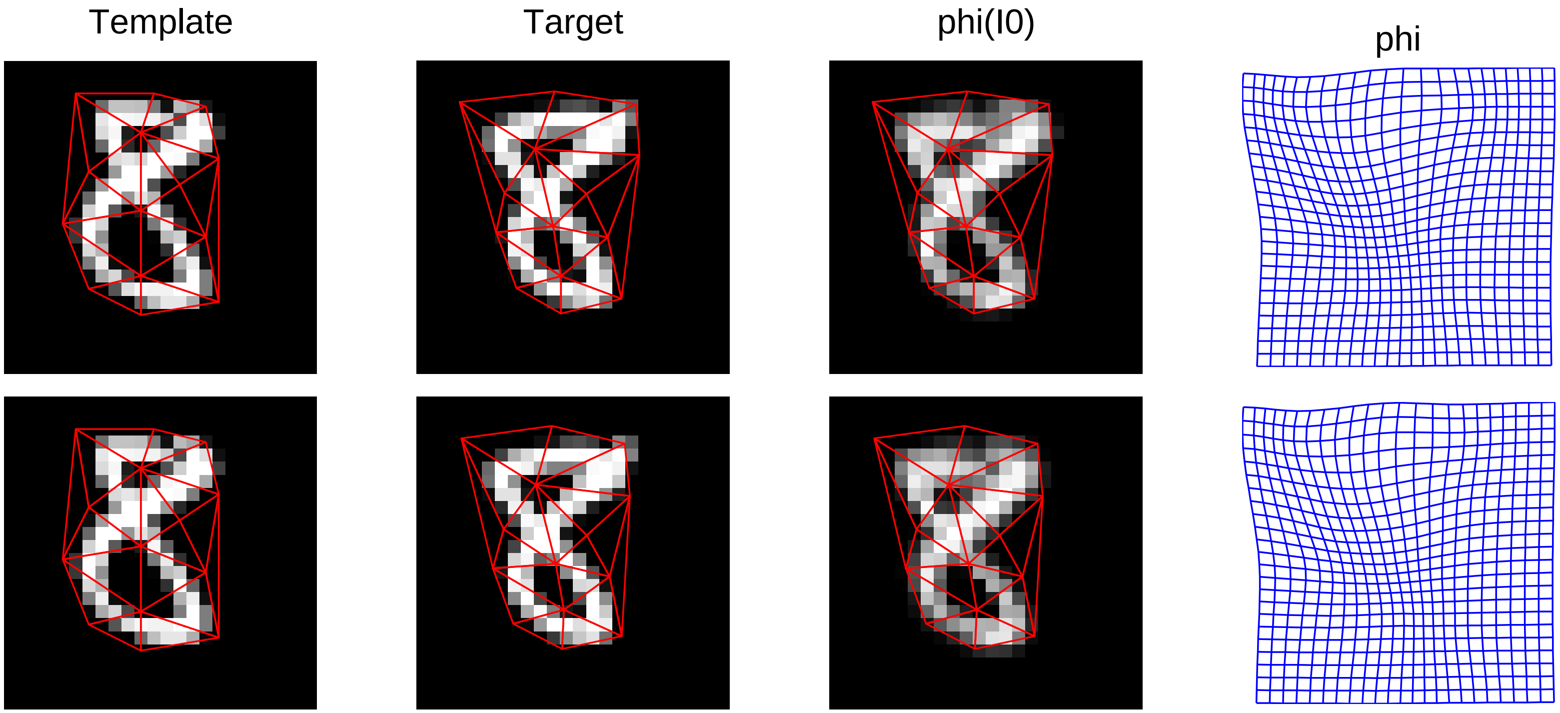
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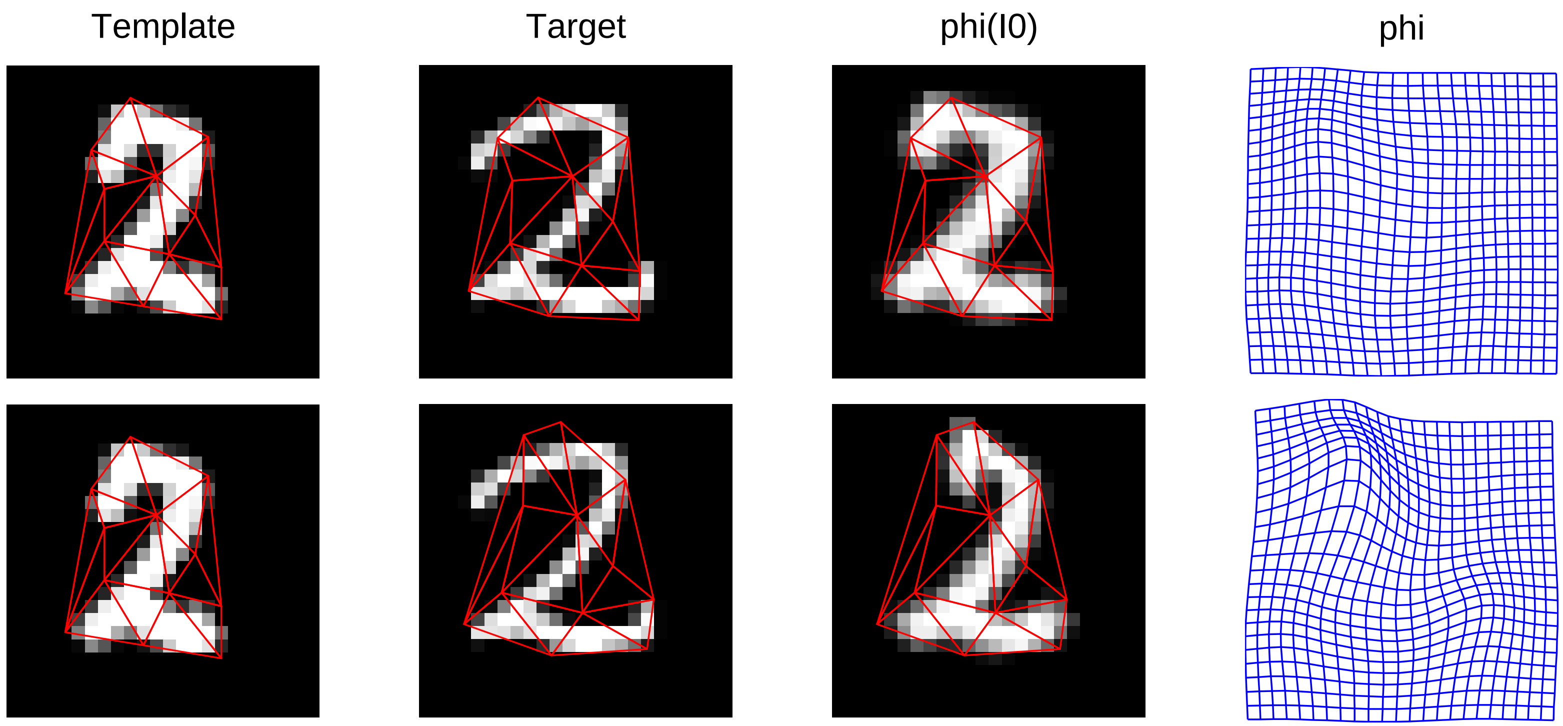
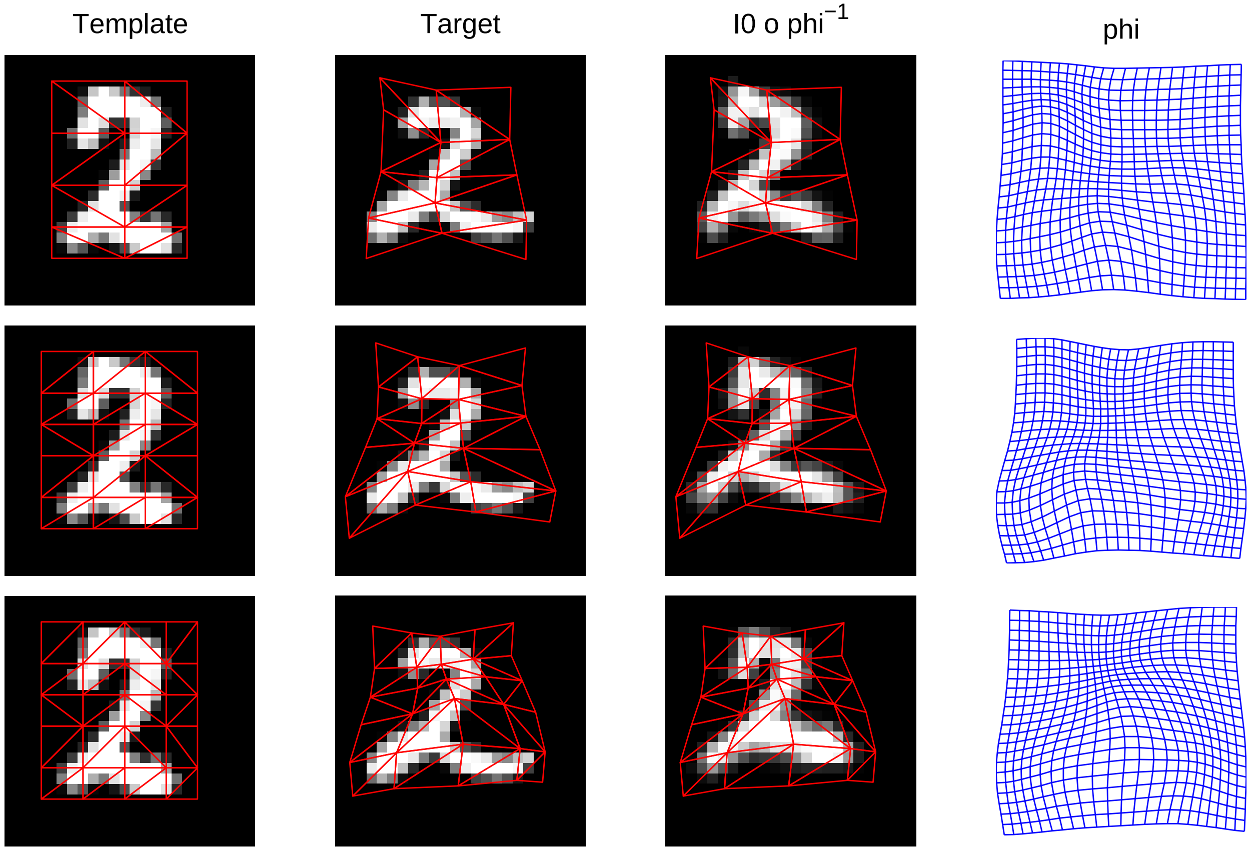
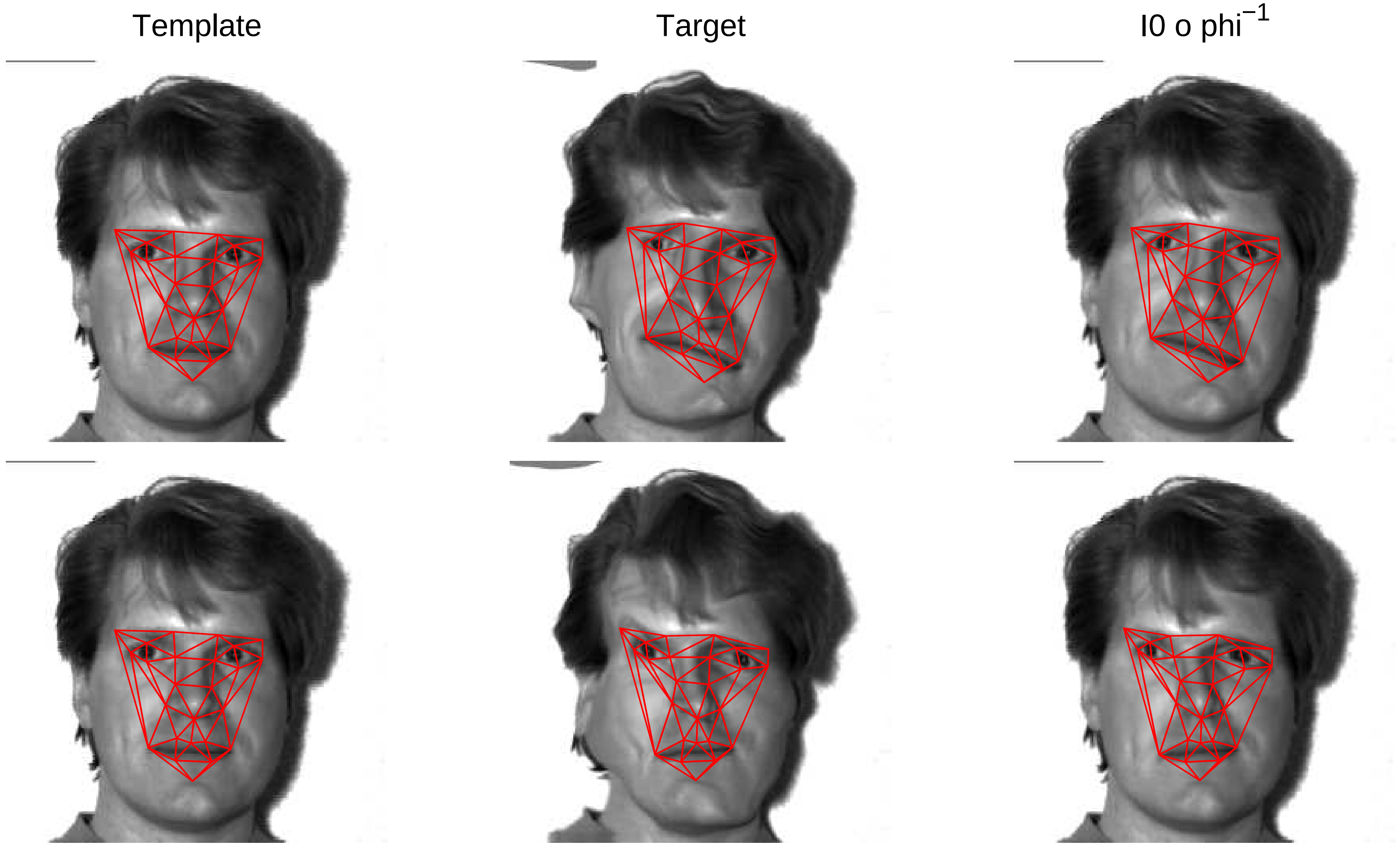
phi


Template
Target
phi(I0)
phi

Common mesh for all images




Template
Target
I0 o phi^{-1}

Limitations of each algorithm

Gradient descent

* The convergence speed

Newton's method

* The initialization point
* The matrix conditionnement.
  Solution Projection on the main singular directions of the matrix before inversion.

Both

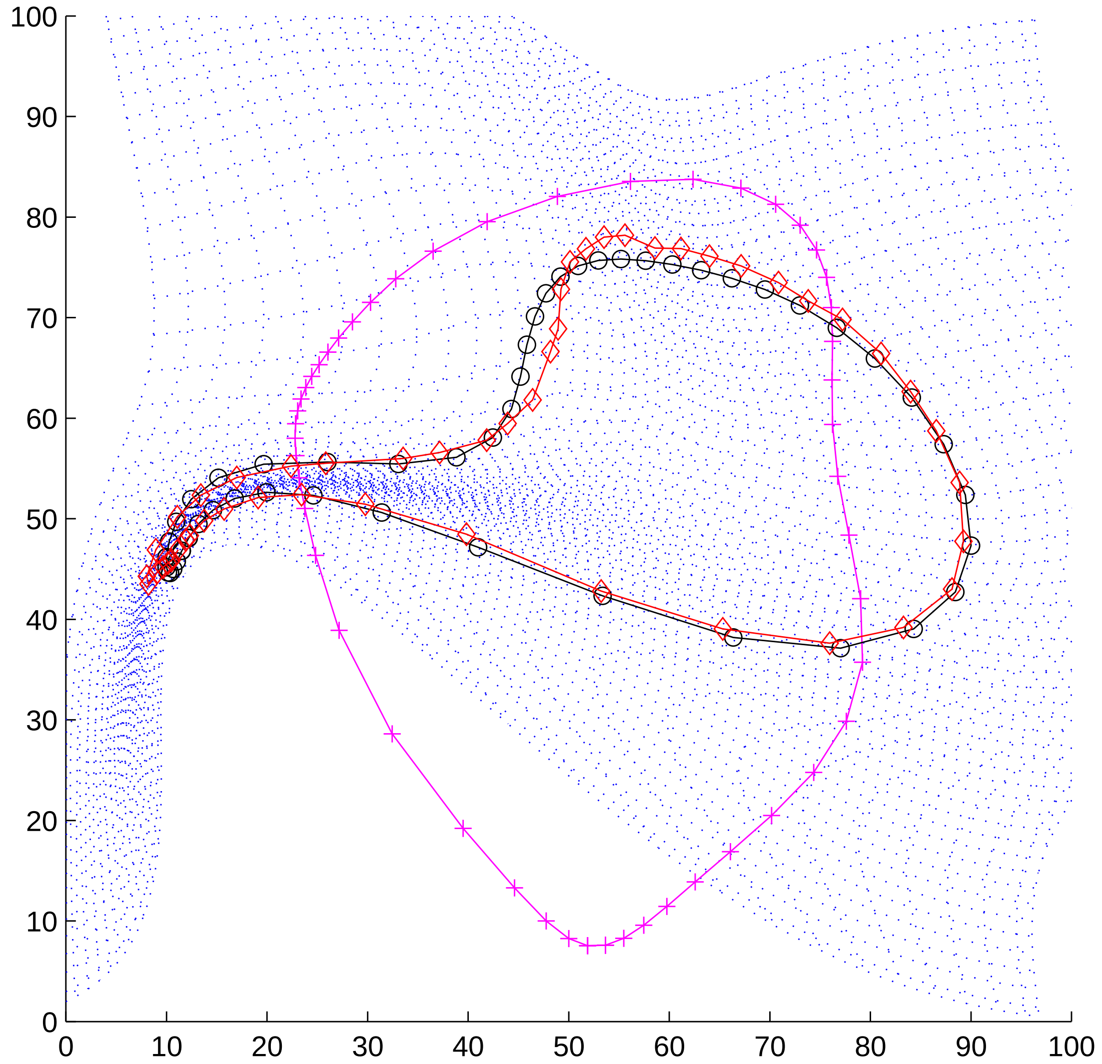
* The triangle consistency to keep an homeomorphic deformation

Remark : Landmark matching

Using the same point of view, we can do landmark matching as well. The energy is given by :

$$E = \frac{1}{2}\int_0^1 ||v_t||_V^2 dt + \lambda \sum_{i=1}^{N} ||q_i^1 - y_i||_{\mathbb{R}^k}^2.$$

the data attachment term derivatives equal $\frac{dg}{dq^1} = 2(q_i^1 - y_i)$ and $\frac{d^2 g}{(dq^1)^2} = 2Id_{N\times d}$, where d is the dimension.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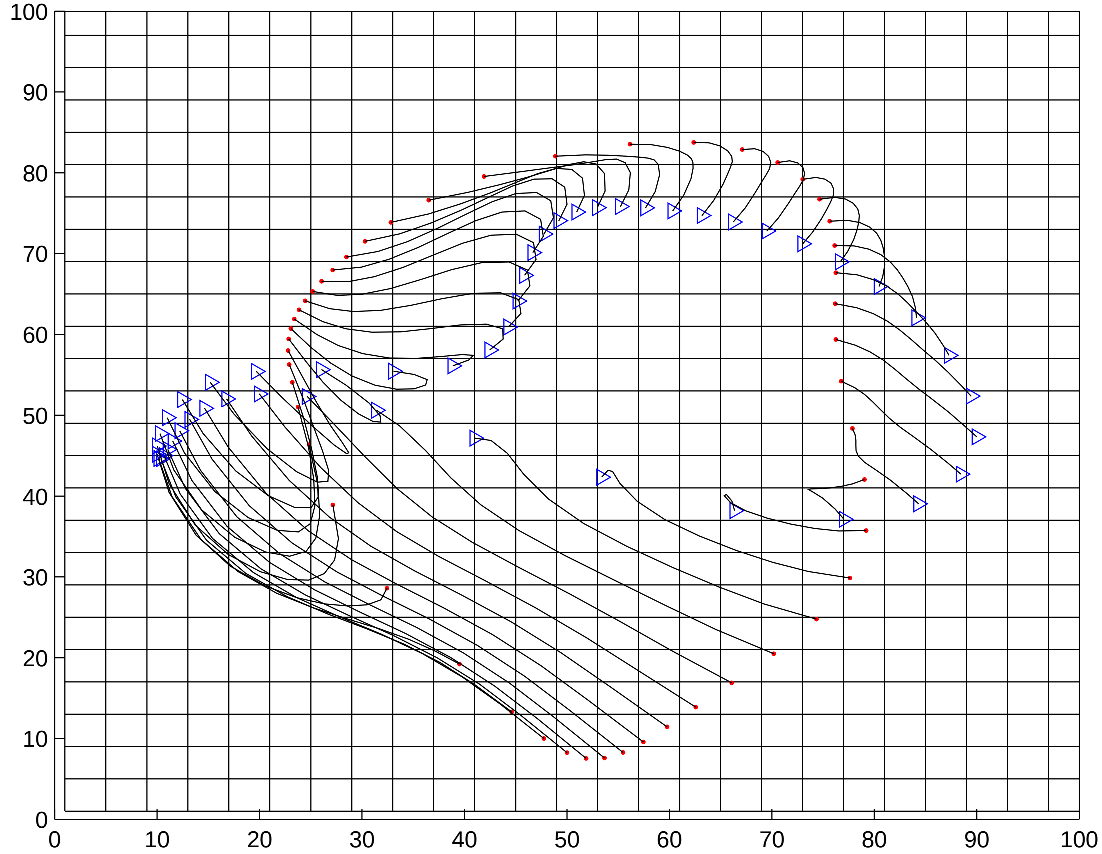
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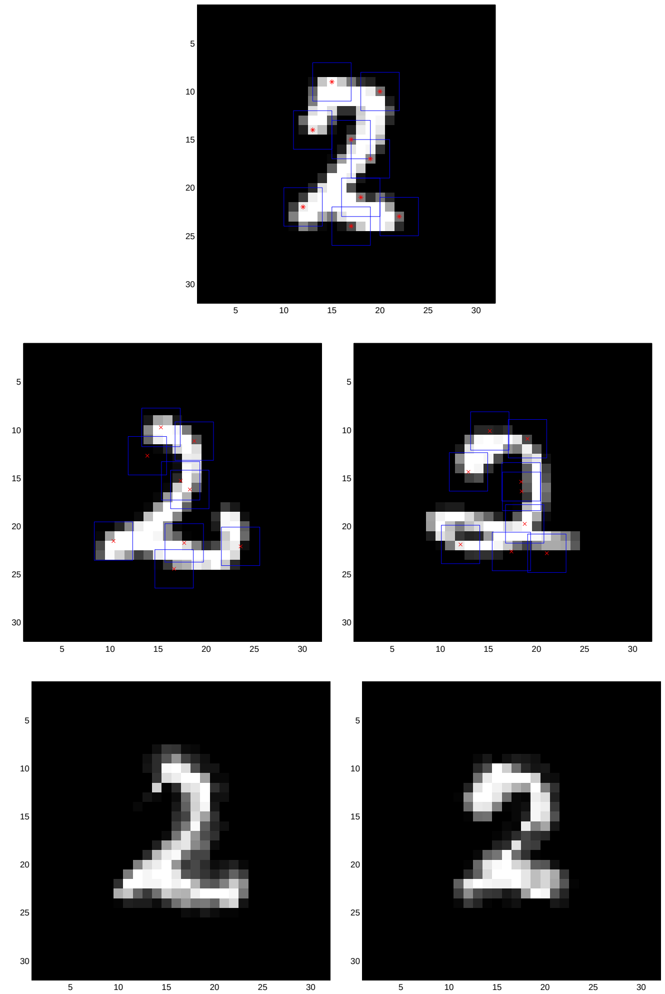
Conclusion

* The triangulation is a way to reduce the system dimension, focusing on landmark evolutions.
* Can be generelized for 3-D images.
* Newton's method has the advantage of speed of convergence.

Automatic landmark detection

Let w be a window. The energy to minimize is :

$$E = \frac{1}{2}\int_0^1 ||v_t||_V^2 dt + \lambda \sum_{i=1}^{N} \int_\Omega |I_0 \circ \phi_1^{-1}(y) - I_1(y)|^2 w(x_i - \phi_1^{-1}(y))dy$$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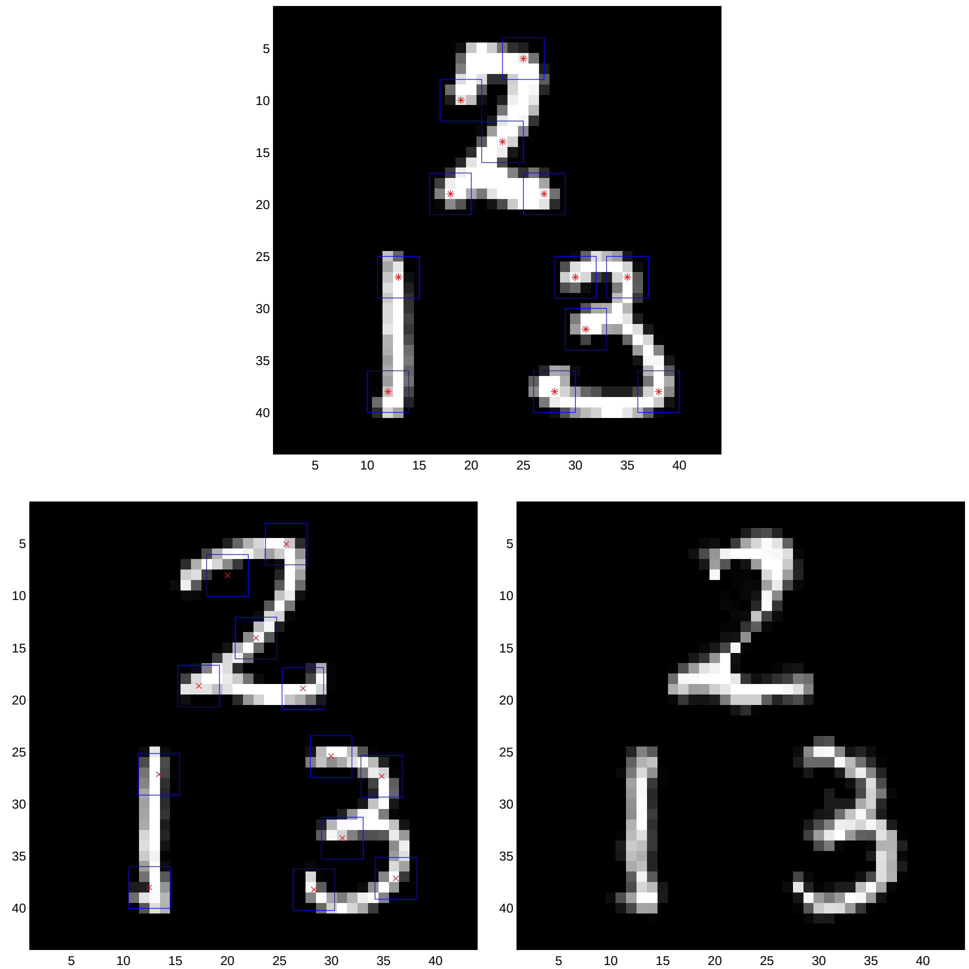
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